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Enhanced feedback robustness against communication channel multiplicative uncertainties via scaled dither

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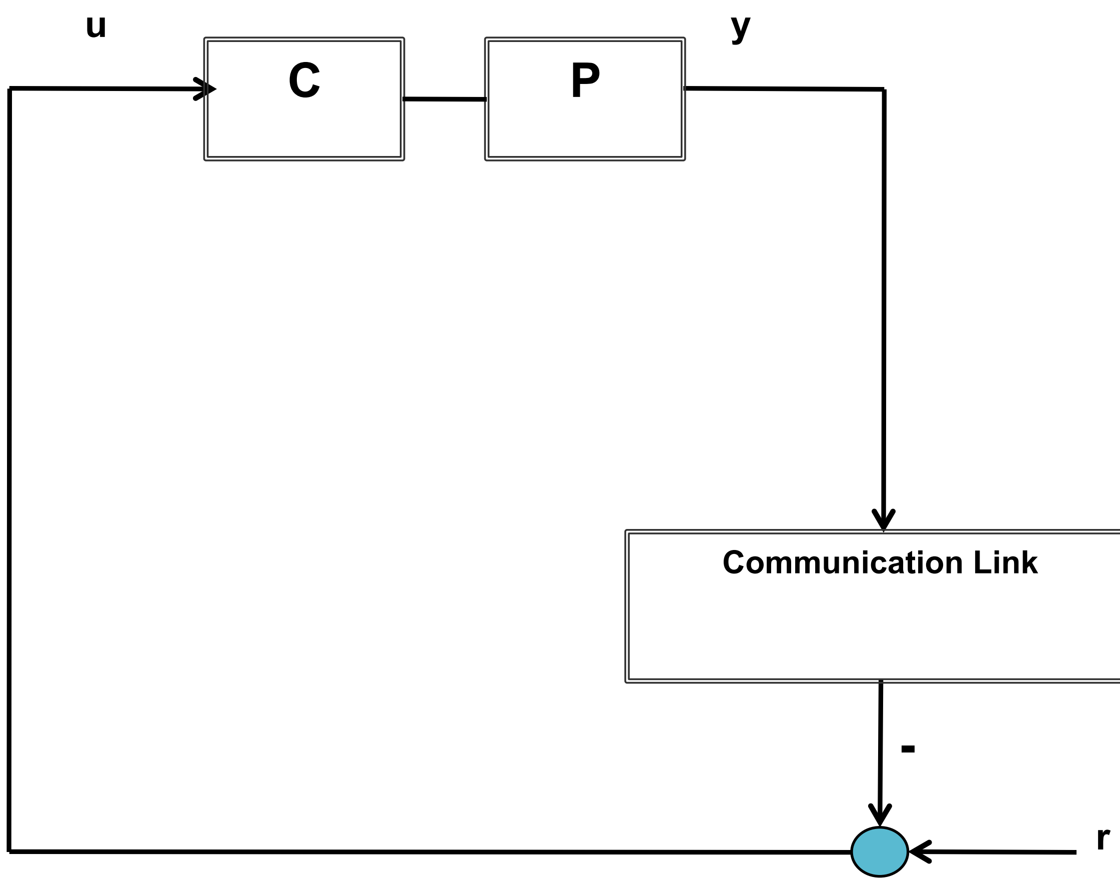
Content

- Problem: communication link in feedback loop (multiplicative gain uncertainties)

- Solution: add a new signal -- scaled dither

- Applications: small grid, vehicle platoons, etc.

Problem



Control feedback over a communication link

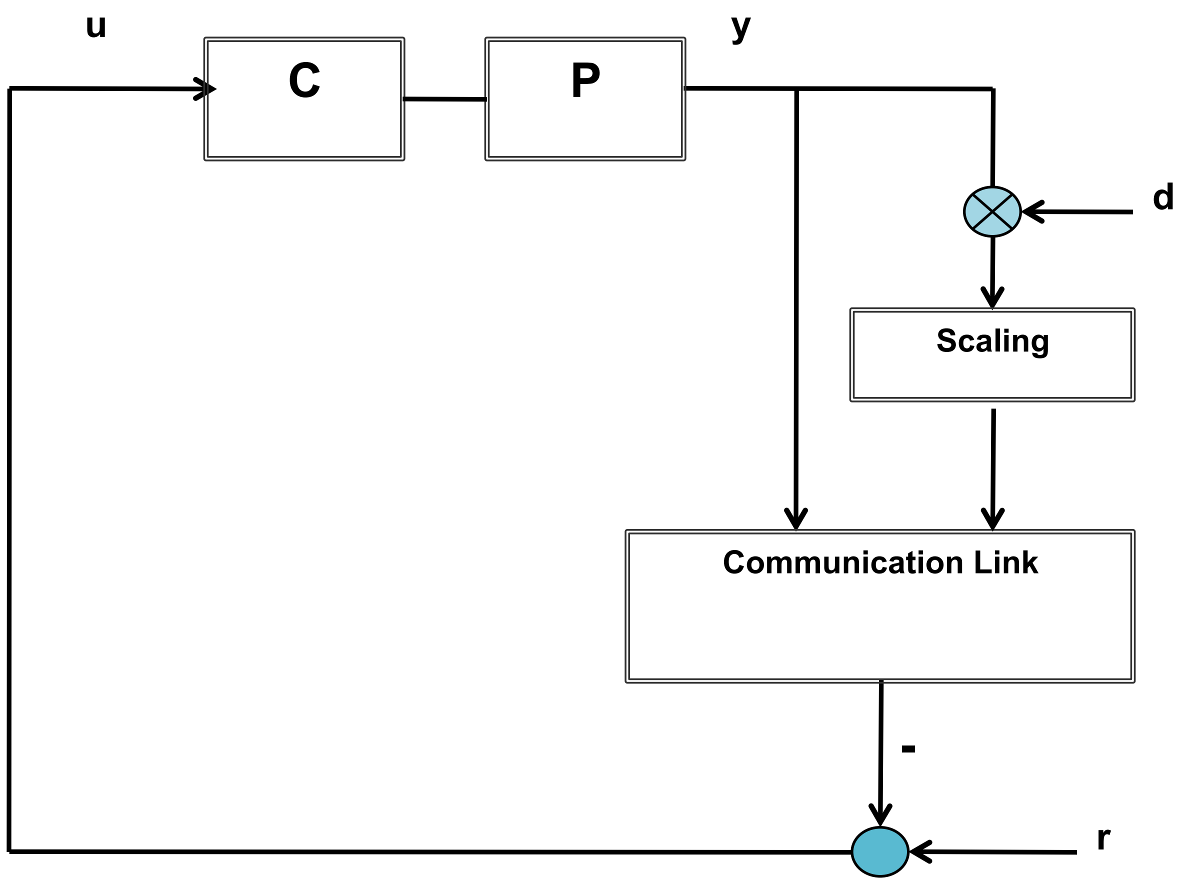
$$x_{k+1} = x_k + \tau_k (Ax_k + Bu_k)$$

$$y_k = Cx_k$$

$$u_k = -g_k y_k$$

$$A - gBC < 0$$

Scaled Dither



Control over a communication link with a scaled dither

$$z_k = y_k + \alpha(\tau_k, y_k)d_k$$

$$\alpha(\tau_k, y_k) = \frac{\gamma}{\sqrt{\tau_k}} y_k$$

$$u_k = -g(y_k + \frac{\gamma}{\sqrt{\tau_k}} y_k d_k) = -g(Cx_k + \frac{\gamma}{\sqrt{\tau_k}} Cx_k d_k)$$

$$x_{k+1} = x_k + \tau_k (A - gBC)x_k + \sqrt{\tau_k}\, g\gamma BCx_k d_k$$

$$\Downarrow \quad \tau_k \geq 0, \tau_k \to 0 (k \to \infty), t_k = \sum_{j=1}^{k} \tau_j \to \infty, (k \to \infty)$$

$$dx = (A - gBC)xdt + g\gamma BCxdw$$

SDE

$$x(t) = x(t_0) + \int_{t_0}^{t} Ax(r)dr + \int_{t_0}^{t} G_1 x(r)dw_1(r) + \int_{t_0}^{t} G_2 dw_2(r)$$

$$dx = Mxdt + G_1 x dw_1 + G_2 dw_2$$

$$\Downarrow \quad G_2 = 0$$

$$dx = Mxdt + G_1 x dw_1$$ one-dimensional Brownian motion

$$\Downarrow$$ scalar case

$$x(t) = e^{(m - \frac{1}{2} g_1^2)t + g_1 w_1} x(0)$$

By the strong law of large numbers, $w_1(t)/t \to 0$ w.p.1.

$$\Rightarrow \quad \limsup_{t\to\infty} \frac{\log x(t)}{t} = m - \frac{1}{2} g_1^2$$

SDE is stable if $m - \frac{1}{2} g_1^2 < 0$ Scalar Case

$$M_c = M - \frac{1}{2} G_1 G_1'$$

Hurwitz, Vector Case

$$\begin{cases} \dot{x} = Ax + Bu, \quad u = -gy = -gCx \\ M = A - gBC \\ G_1 = g\gamma BC \end{cases}$$

$$\Downarrow$$

$$= A - gBC - \frac{1}{2} g^2 \gamma^2 BCC'B'$$

Scaled Dither

$$M_c(g_0) = a - g_0 bc - \frac{1}{2} g_0^2 \gamma^2 b^2 c^2$$

$$\Downarrow \quad g = g_0 bc$$

$$M_c(g) = a - g - \frac{1}{2} g^2 \gamma^2$$

$g \in \Omega$

$$\Downarrow \quad \lambda_1 = \frac{-1 + \sqrt{1 + 2a\gamma^2}}{\gamma^2}, \quad \lambda_2 = \frac{-1 - \sqrt{1 + 2a\gamma^2}}{\gamma^2}$$

Theorem 1 *(1) If* $a < 0$ *, then if* $\gamma > 1/2|a|$ *, the closed-loop system is stable for all* g

(2) If $a \geq 0$ *, then for any given* γ *, the closed-loop system is stable for all*
$g \in \Omega = (-\infty, \lambda_1) \cap (\lambda_2, \infty)$

Explicit Bounds on Relative Gain Robustness

$$\Omega = \{g = 1+\delta : |\delta| \le \bar{\delta}, |1+\delta| \ge \mu > 0\}$$

Theorem 3 *Suppose that the nominal*

system $M_c(1) = a - 1 < 0$ *. If* $\gamma > \sqrt{\frac{2\bar{\delta}}{\mu^2}}$

the closed-loop system is robustly stable

for all $\delta \in \Omega$

$$\delta | \le < \frac{\mu^2 \gamma^2}{2} \le \frac{(1+\delta)^2 \gamma^2}{2}$$

$$-\delta - \frac{(1+\delta)^2 \gamma^2}{2} < 0$$

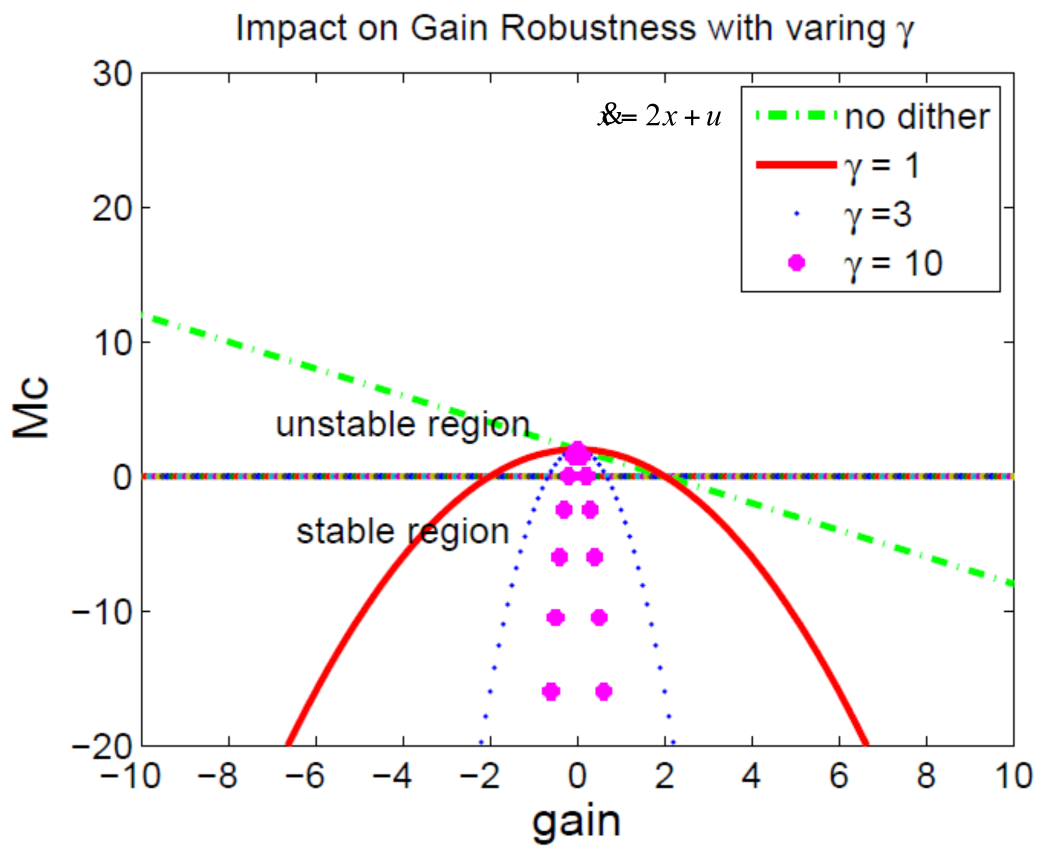


Figure 5: Robustness enhencement with varing γ

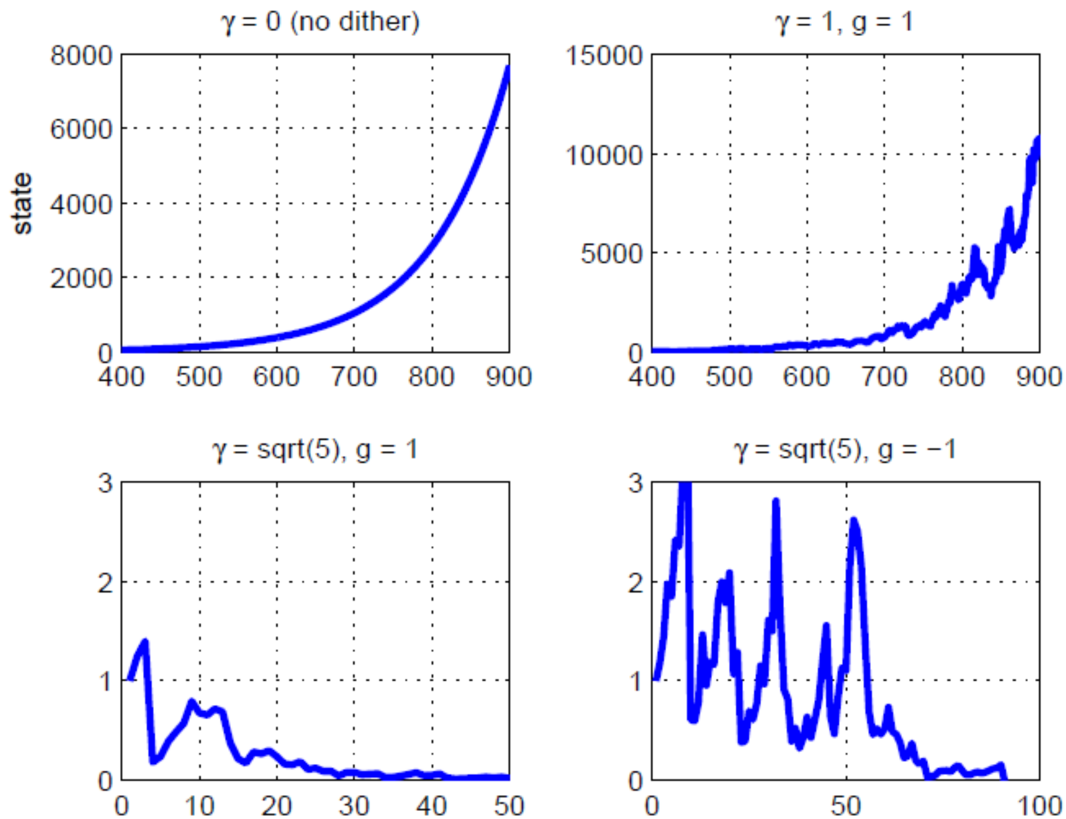


Figure 6: Norminal Stability with varing γ and g

Theorem 6 *Under the same assumption as*

in Theorem 3 $z_k = g_0 \frac{\gamma x_k}{\sqrt{\tau_k}} d_k, g = g_0 bc$ *and*

$$\gamma > \frac{\sqrt{2|a|}}{\mu}$$

The system is robustly stable for all $\delta \in \Omega$

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) = \begin{bmatrix} 0 & 1 \\ a_1 & a_2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ y(t) = Cx(t) = [c_1 \; c_2] x \end{cases}$$

l

$\Downarrow$ $u = ly$ deterministic framework

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ a_1 - lc_1 & a_2 - lc_2 \end{bmatrix} x$$

$a_1 = 0, a_2 = 0, c_1 = 1, c_2 = -1$

No Z tunable

Second Order System
Dither

$$u_k = -ly_k - \gamma y_k d_k$$

$$A_c = A - lBC - \frac{1}{2}\gamma^2 BC(BC)^T = \begin{bmatrix} 0 & 1 \\ a_1 - lc_1 & \overset{c_1}{a_2} - lc_2 - \frac{\gamma^2}{2}(c_1^2 + c_2^2) \end{bmatrix}$$

$$a_1 - lc_1 < 0, \quad a_2 - lc_2 - \frac{\gamma^2}{2}(c_1^2 + c_2^2) < 0$$

always be stable if $c_1 \neq 0$

General Case

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -0.5 & -2 & -5 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, C = [1, 1.5, 2] \qquad \gamma = 20$$

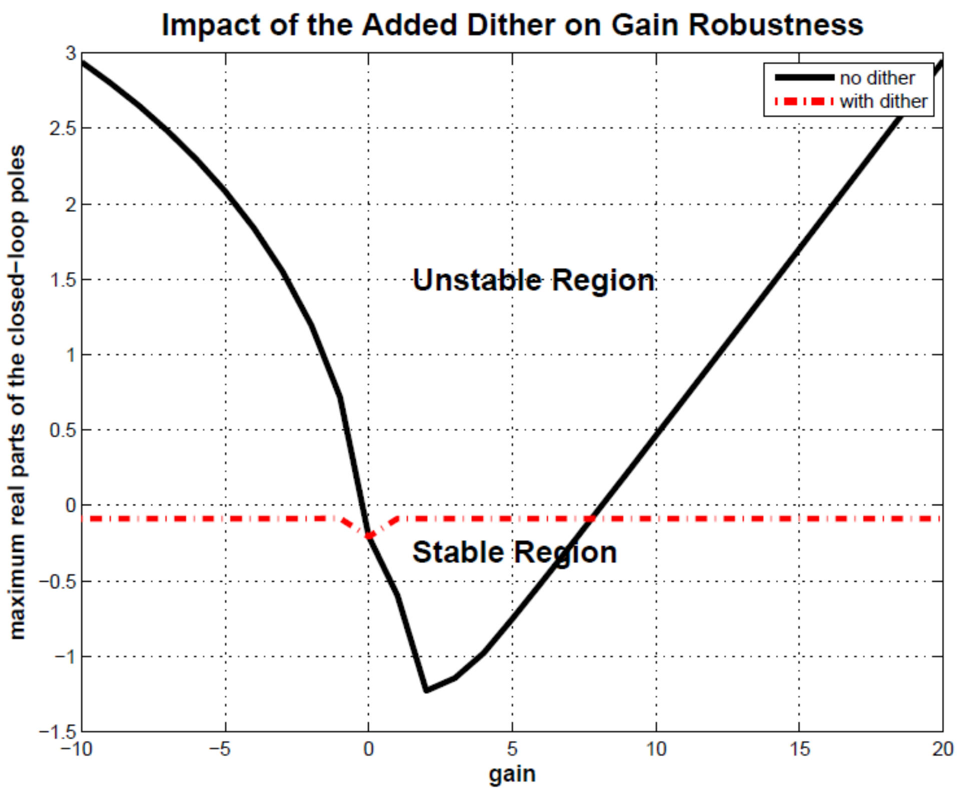


Figure 7: Robustness again gain uncertainties

$$M_c^0 = A - gBC = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & -4 & -6 & -4 \end{bmatrix}$$

M_c^0 has eigenvalues $-1, -1, -1, -1$

$$M_c^1 = M_c^0 - 0.5\gamma^2 (Bc)(BC)' == \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & -4 & -6 & -54 \end{bmatrix}$$

$\gamma = 10$ Not stable

Consensus Control

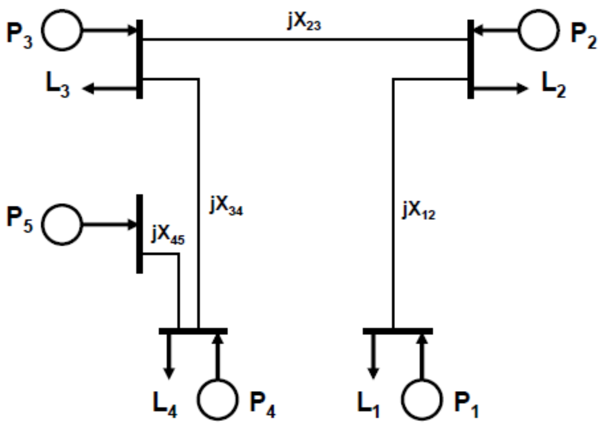


Figure 8: A grid of five buses

$$g = \{(1,2),(2,1),(2,3),(3,2),(3,4),(4,3),(4,5),(5,4)\}.$$

$$x_n = [x_n^1, \ldots, x_n^r]' = [P^1, P^2, P^3, P^4, P^5]'$$

$$x_{n+1} = x_n + u_n \qquad x_0 = [0.1, 0.2, 0.3, 0.4, 0]'$$

$$\hat{x}_n^{ij} = x_n^j + d_n^{ij}$$

$$\tilde{x}_n = H_1 x_n + d_n = [\hat{x}^{12}, \hat{x}^{21}, \hat{x}^{23}, \hat{x}^{32}, \hat{x}^{34}, x^{43}, x^{45}, \hat{x}^{54}]'$$

$$\delta_n = H_2 x_n - \tilde{x}_n = H_2 x_n - H_1 x_n - d_n = H x_n - d_n,$$

$$u_n = -\mu_n H' G \delta_n = -\mu_n H' G (H x_n - d_n) = -\mu_n (H' G H x_n - H' G d_n) = \mu_n (M x_n + W d_n)$$

$$x_{n+1} = x_n + \mu_n M x_n + \mu_n W d_n$$

Consensus Control without Gain Uncertainties

$$H_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}; \qquad H_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

$$H = H_2 - H_1 = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}$$

$$G = \mathrm{diag}[0.3, 0.3, 0.5, 0.7, 0.9, 0.9, 1, 1]$$

$$M = -H'GH = \begin{bmatrix} -0.6 & 0.6 & 0 & 0 & 0 \\ 0.6 & -1.8 & 1.2 & 0 & 0 \\ 0 & 1.2 & -3 & 1.8 & 0 \\ 0 & 0 & 1.8 & -3.8 & 2 \\ 0 & 0 & 0 & 2 & -2 \end{bmatrix}$$

$$W = H'G = \begin{bmatrix} 0.3 & -0.3 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.3 & 0.3 & 0.5 & -0.7 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.5 & 0.7 & 0.9 & -0.9 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.9 & 0.9 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}$$

$$eig(M) = -6.0125, -3.2432, -1.5016, -0.4426, 0$$

Consensus Control without Gain Uncertainties

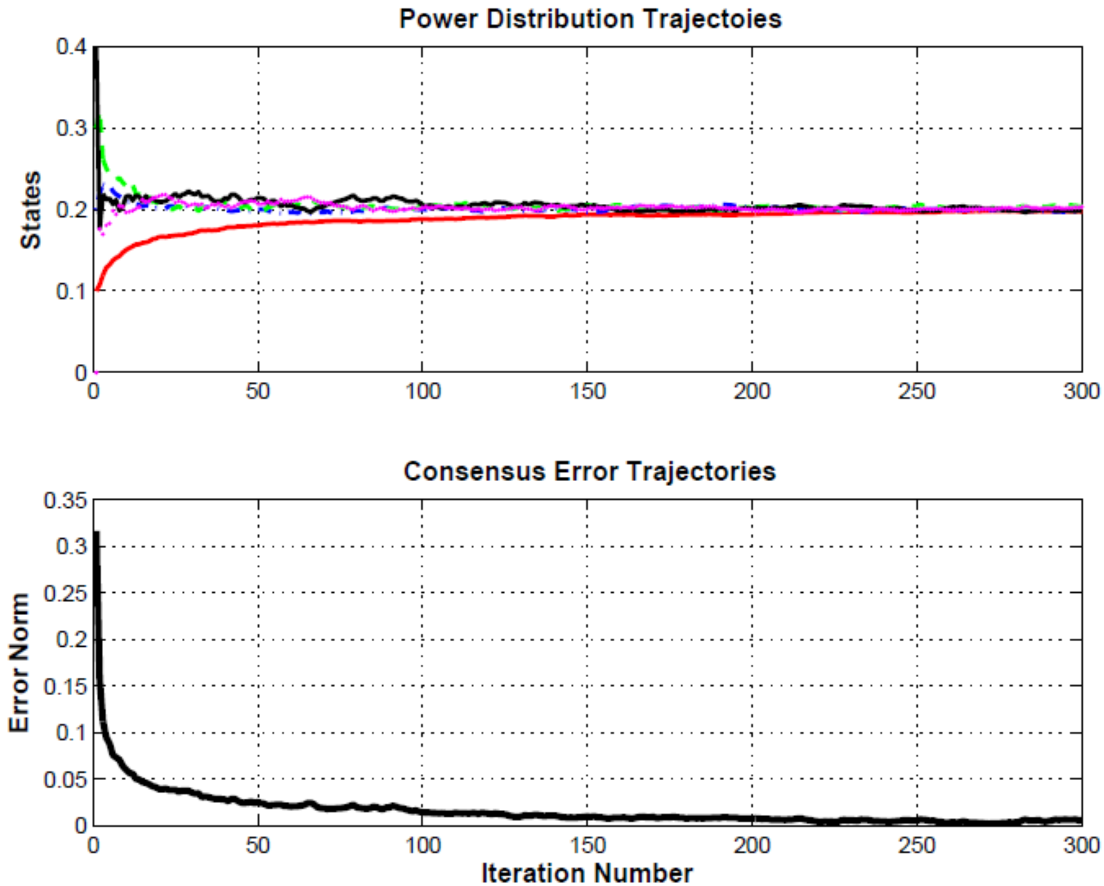
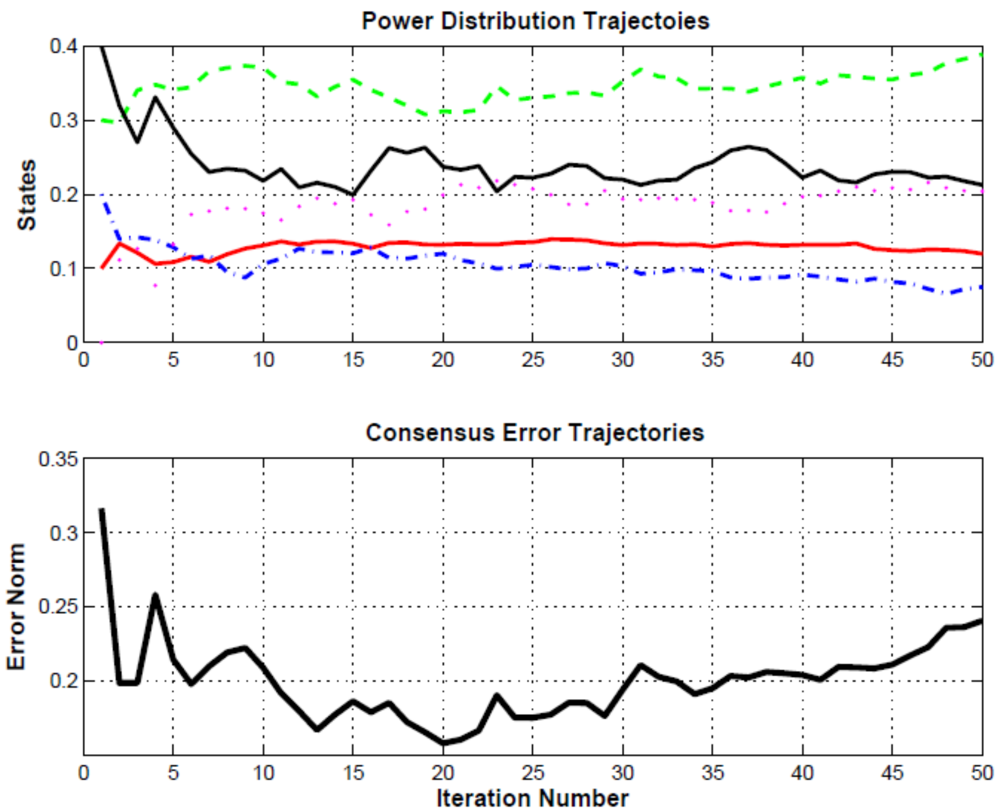


Figure 9: Power flow control under positive link gains

Consensus Control with Gain Uncertainties



$$G = \mathrm{diag}[0.3, 0.3, 0.5, -0.7, -0.9, 0.9, 1, 1]$$

$$M = -H'GH = \begin{bmatrix} -0.6 & 0.6 & 0 & 0 & 0 \\ 0.6 & -0.4 & -0.2 & 0 & 0 \\ 0 & -0.2 & 0.2 & 0 & 0 \\ 0 & 0 & 0 & -2 & 2 \\ 0 & 0 & 0 & 2 & -2 \end{bmatrix}$$

Figure 10: Power flow control under perturbed link gains

$$d_k = \gamma x_k^j / \sqrt{\tau_k}$$

Assumption: (1) All link gains are non-zero $g_{ij} \neq 0$. (2) The network topology g contains a full tree.

$$dx = Mxdt + G_1 x dw_1 + W dw_2$$

Theorem 15 Q *is negative semi-definite and has rank r−1. Hence, the eigenvalues of Q are all negative, except one which is 0.*

$$M = -H'GH, W = H'G, G_1 = \gamma H'GH_1$$

$$Q = M - \frac{1}{2} G_1 G_{1'} = -H'GH - \frac{\gamma^2}{2} H'GH_1 H_{1'} GH$$

Consensus Control with Scaled Dither

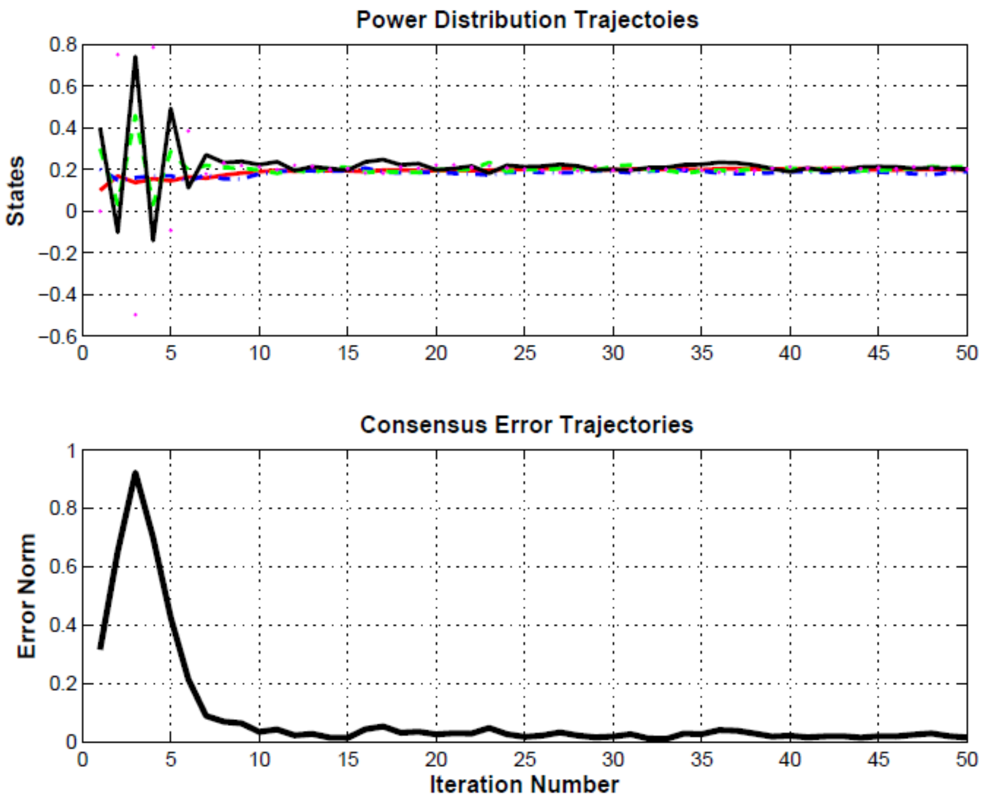


Figure 11: Power flow control under perturbed link gains but with a scaled dither added to each observation link

$$\sigma^2 = 9$$

$$Q = \begin{bmatrix} -1.41 & 0.465 & 0.945 & 0 & 0 \\ 0.465 & -2.65 & 4.21 & -2.025 & 0 \\ 0.945 & 4.21 & -14.47 & 5.265 & 4.05 \\ 0 & -2.025 & 5.265 & -10.19 & 6.95 \\ 0 & 0 & 4.05 & 6.95 & -11 \end{bmatrix}$$

$$eig(Q) = \text{-19.3659, -17.1275, -2.0336, -1.1931, 0}$$

(1). Contributions

(2). Future Works

Thank You!