Impedance Matching Equation: Developed Using Wheeler's Methodology

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Outline

1. Background Information
2. The Impedance Matching Equation
3. The Bode and Fano Impedance Matching Equations
4. Wheeler’s Single- and Double-Tuning Equations
5. Conversion of Wheeler’s Equations to the Original Impedance Matching Equation
6. Development of the final form for the Impedance Matching Equation
7. A note on Triple-Tuned Impedance Matching

Background Information

1940s

- Wheeler develops impedance matching principles
- A Wheeler designed double-tuned impedance-matched IFF antenna played a critical role in WW II
- Bode and Fano publish their work on impedance matching

1950

- Wheeler publishes Report 418, a tutorial on impedance matching that features the reflection chart as a primary tool
- For single- and double-tuned impedance matching, it presents three equations that quantify impedance-matching bandwidth limitations related to a specified maximum reflection magnitude
- Based on the works of Bode and Fano, it quantifies the law of diminishing returns for impedance matching circuits beyond double tuning

1973

- Wheeler's three equations are converted to the original Impedance Matching Equation

2004

- Using MATCAD to solve Fano's equations, the final version of the Impedance Matching Equation was developed

Impedance-Matching Equation

$$\mathbf{B_n}(\Gamma) = \frac{1}{Q} \frac{1}{b_n \sinh\left(\frac{1}{a_n}\ln\left(\frac{1}{\Gamma}\right)\right) + \frac{1-b_n}{a_n}\ln\left(\frac{1}{\Gamma}\right)}$$

Assumes Lumped-Element Circuits Exact for n = 1, 2, and ∞

QB_n Error < 0.1% for $\Gamma > 0.10$ (Max VSWR > 1.2)

B_n = Maximum fractional impedance-matching bandwidth

$B_n = (f_H - f_L)/f_0$

f_0 = Resonant frequency = $\sqrt{f_H f_L}$

Q = Antenna Q (Ratio of reactive power to radiated and dissipated power}

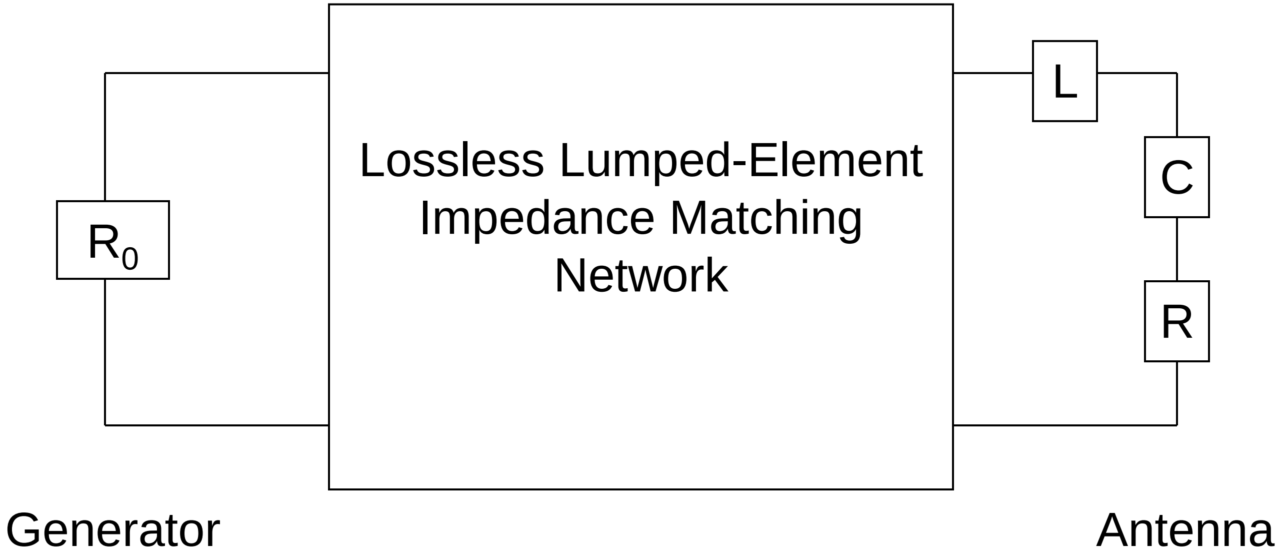
Γ = Maximum reflection magnitude within B_n

n = Number of tuned stages in the impedance matching circuit

n	a_n	b_n		n	a_n	b_n
1	1	1		6	2.838	0.264
2	2	1		7	2.896	0.209
3	2.413	0.678		8	2.937	0.160
4	2.628	0.474				
5	2.755	0.347		∞	π	0

Bode Impedance Matching Equation

(Hendrik W. Bode)

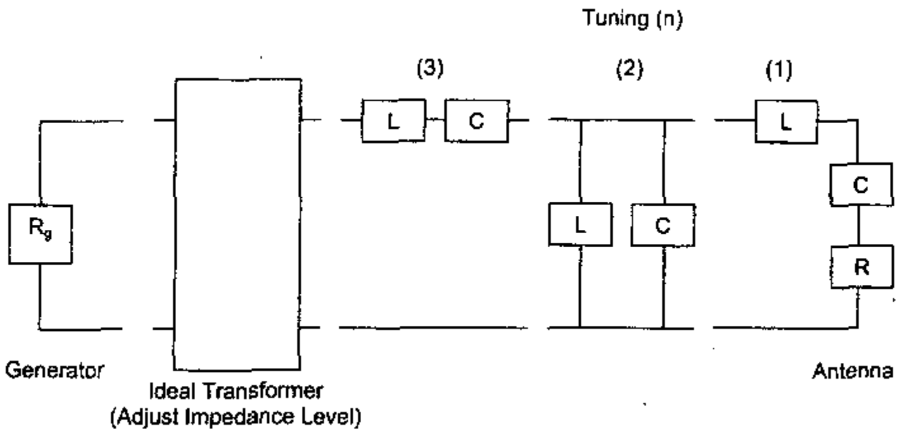


$$B = \frac{1}{Q}\frac{\pi}{\ln\left(\frac{1}{\Gamma}\right)} \qquad Q = \frac{\omega_0 L}{R}$$

B = Theoretical maximum fractional bandwidth for specified maximum reflection magnitude

Fano's Impedance Matching Equations

(Robert M. Fano)



n tuned stages
Alternate - series and parallel
All stages tuned to f_0
n = 1 is the tuned antenna

Γ
n $\longrightarrow$

$$\Gamma = \frac{\cosh(nb)}{\cosh(na)}$$

$$\frac{\tanh(na)}{\cosh(a)} = \frac{\tanh(nb)}{\cosh(b)}$$

$$\frac{2\sin\left(\frac{\pi}{2n}\right)}{\sinh(a) - \sinh(b)} = QB$$

$\longrightarrow$ $QB_n(\Gamma)$

NOTE: The Impedance Matching Equation is a closed-form approximate solution for the Fano Impedance Matching Equations

The Bode-Fano Equation

Fano showed that in the limit case of n = ∞

$$B_{\infty} = \frac{1}{Q} \frac{\pi}{\ln\left(\frac{1}{\Gamma}\right)}$$

We Started in 1973 With Wheeler's Three Equations for a Resonant Antenna

1950 Wheeler Lab Report 418

1. $QB = \tan(\varphi_{EB})$ $\quad \varphi_{EB} = \text{Magnitude of impedance phase at edge-band frequencies}$

2. $\Gamma_1 = \tan\left(\frac{\varphi_{EB}}{2}\right)$ (Optimum Single Tuning)

3. $\Gamma_2 = \Gamma_1^{\,2}$ (Optimum Double Tuning)

Wheeler's First Equation

Wheeler's Small Resonant Antenna
Lumped-Element RLC Circuit
Example: Small Electric Dipole
Capacitor resonated with series L

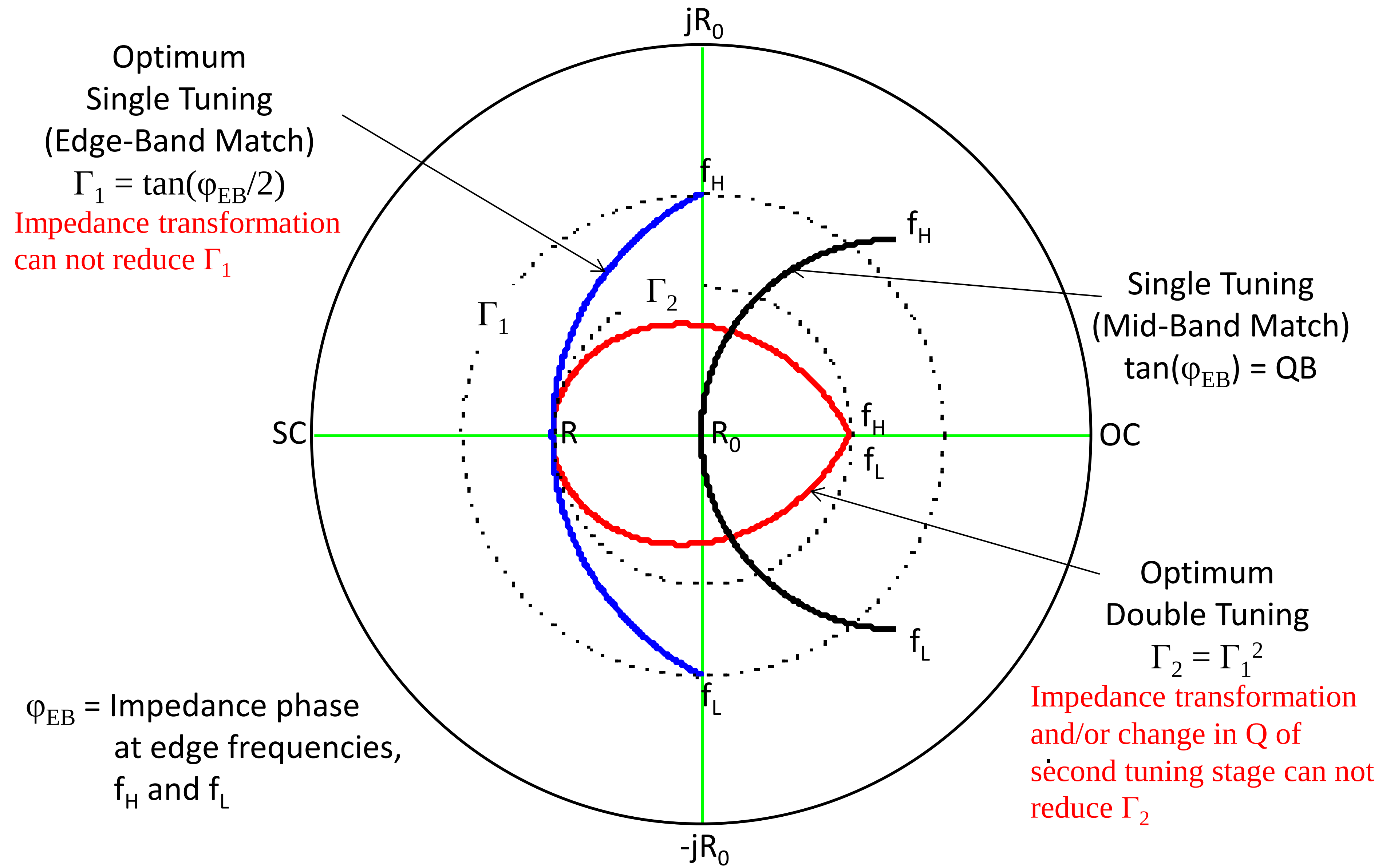
$$Z_{EB} = R + j\frac{1}{\omega_0 C}\left(\frac{f_H}{f_0} - \frac{f_0}{f_H}\right)$$

$$Z_{EB} = R\left(1 + j\frac{1}{\omega_0 CR}\left(\frac{f_H}{f_0} - \frac{f_L}{f_0}\right)\right)$$

$$Z_{EB} = R(1 + jQB) = R \cdot \exp(j\varphi_{EB})$$

$$\tan(\varphi_{EB}) = QB$$

Wheeler's Optimum Single- and Double-Tuned Impedance Matching (Proof by Inspection)



φ_{EB} = Impedance phase at edge frequencies, f_H and f_L

Single Tuning: Derivation of $\left|\Gamma_{EB}\right| = \tan\left(\frac{\varphi_{EB}}{2}\right)$

From Reflection Chart $R_0 = 1$ $\longrightarrow$ $Z_{EB} = e^{j\varphi_{EB}}$

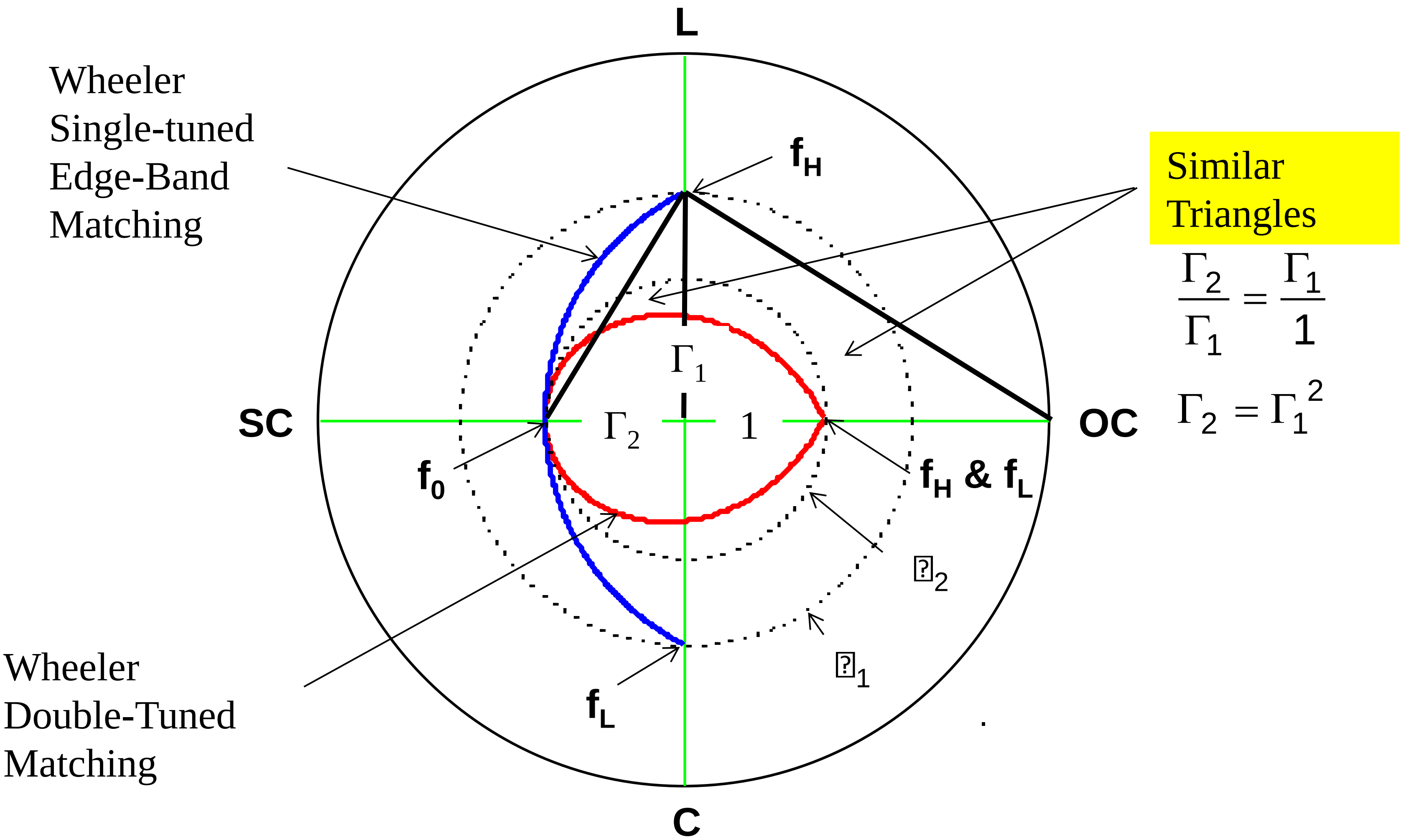
$$\Gamma_{EB} = \frac{e^{j\varphi_{EB}} - 1}{e^{j\varphi_{EB}} + 1}$$

$$\Gamma_{EB} = \frac{\cos(\varphi_{EB}) + j\sin(\varphi_{EB}) - 1}{\cos(\varphi_{EB}) + j\sin(\varphi_{EB}) + 1}$$

$$\Gamma_1 = \left|\Gamma_{EB}\right| = \frac{\sqrt{\cos^2(\varphi_{EB}) - 2\cos(\varphi_{EB}) + 1 + \sin^2(\varphi_{EB})}}{\sqrt{\cos^2(\varphi_{EB}) + 2\cos(\varphi_{EB}) + 1 + \sin^2(\varphi_{EB})}}$$

$$\Gamma_1 = \frac{\sqrt{1 - \cos(\varphi_{EB})}}{\sqrt{1 + \cos(\varphi_{EB})}} = \tan\left(\frac{\varphi_{EB}}{2}\right)$$

Derivation of $\Gamma_2 = \Gamma_1^2$



$$\frac{\Gamma_2}{\Gamma_1} = \frac{\Gamma_1}{1}$$

$$\Gamma_2 = \Gamma_1^2$$

In 1973 we converted Wheeler's three equations for a resonant antenna to a single equation

1. $QB = \tan\left(\varphi_{EB}\right)$ $\quad \varphi_{EB} = \text{Impedance phase at edge frequency}$
2. $\Gamma_1 = \tan\left(\varphi_{EB}/2\right)$ (Single Tuning)
3. $\Gamma_2 = \Gamma_1^{\,2}$ (Double Tuning)

Single Tuning : $\tan(\varphi) = \dfrac{2\tan(\varphi/2)}{1-\tan^2(\varphi/2)}$ $\quad QB_1 = \dfrac{2\Gamma_1}{1-\Gamma_1^{\,2}}$

Double Tuning : $\quad QB_2 = \dfrac{2\sqrt{\Gamma_2}}{1-\Gamma_2}$

Wheeler's Equation:
Single tuning, n = 1
Double tuning, n = 2

$$B_n(\Gamma) = \frac{1}{Q}\,\frac{2\Gamma^{\frac{1}{n}}}{1-\Gamma^{\frac{2}{n}}}$$

1973 Continued

- At this point we had an explicit expression that related B, Q, Γ, and n for single- and double-tuned impedance matching
- We were aware of the Bode and Fano results
- Wheeler clearly defined the law of diminishing returns for added stages beyond double tuning
- One remaining question was: How much bandwidth increase can be achieved with triple tuning over that of double tuning?

1973 Continued

Wheeler's Equation: $$QB_n = \frac{2\Gamma^{\frac{1}{n}}}{1-\Gamma^{\frac{2}{n}}}$$

$$\Downarrow$$

$$QB_1 = \frac{2\Gamma}{1-\Gamma^2} = \frac{2}{\frac{1}{\Gamma}-\Gamma} = \frac{2}{e^{\ln\left(\frac{1}{\Gamma}\right)} - e^{-\ln\left(\frac{1}{\Gamma}\right)}} = \frac{1}{\sinh\left(\ln\left(\frac{1}{\Gamma}\right)\right)}$$

$$QB_2 = \frac{2\sqrt{\Gamma}}{1-\Gamma} = \frac{2}{\frac{1}{\sqrt{\Gamma}}-\sqrt{\Gamma}} = \frac{2}{e^{\frac{1}{2}\ln\left(\frac{1}{\Gamma}\right)} - e^{-\frac{1}{2}\ln\left(\frac{1}{\Gamma}\right)}} = \frac{1}{\sinh\left(\frac{1}{2}\ln\left(\frac{1}{\Gamma}\right)\right)}$$

$$\Downarrow$$

$$B_n = \frac{1}{Q}\frac{1}{\sinh\left(\frac{1}{a_n}\ln\left(\frac{1}{\Gamma}\right)\right)} \approx \frac{1}{Q}\frac{a_n}{\ln\left(\frac{1}{\Gamma}\right)} \text{ for } \Gamma > \frac{1}{3}$$

$$a_1 = 1, \text{ and } a_2 = 2$$

1973 Continued

Bode - Fano Equation

$$B_\infty = \boxed{\frac{1}{Q}\frac{\pi}{\ln\left(\frac{1}{\Gamma}\right)}} \qquad a_\infty = \pi$$

For all n and $\Gamma > 1/3$:

$$\boxed{\text{Is } B_n \approx \frac{1}{Q}\frac{a_n}{\ln\left(\frac{1}{\Gamma}\right)} ???}$$

Knew that $a_1 = 1$, $a_2 = 2$, and $a_\infty = \pi$

Ref.: L.B.W. Jolley, "Summation of Series," Dover, New York, (410), p. 76, 1961

$$1+\frac{1}{3}+\frac{1}{5}\left(\frac{2}{3}\right)^2+\frac{1}{7}\left(\frac{2}{3}\frac{4}{5}\right)^2+\ldots\ldots\ldots\infty=\frac{\pi}{2}$$

$$1+1+\frac{1}{3}+\frac{1}{3}+\frac{1}{5}\left(\frac{2}{3}\right)^2+\frac{1}{5}\left(\frac{2}{3}\right)^2+\frac{1}{7}\left(\frac{2}{3}\frac{4}{5}\right)^2+\frac{1}{7}\left(\frac{2}{3}\frac{4}{5}\right)^2+\ldots\ldots\ldots\infty=\pi$$

$$a_n = \sum_{k=1}^{n} s_k \quad a_1 = 1 \quad a_2 = 2 \quad a_3 = 2.333 \quad a_4 = 2.667 \quad a_5 = 2.756 \ \ldots \ a_\infty = \pi$$

1973 Impedance-Matching Equation (Original Equation)

$$B_n(\Gamma) \approx \frac{1}{Q} \frac{1}{\sinh\left(\frac{1}{a_n} \ln\left(\frac{1}{\Gamma}\right)\right)}$$

Exact for n = 1 and 2

Approximate for $\Gamma > 1/3$, and n > 2

n	**a_n**		**n**	**a_n**
1	1		6	2.84
2	2		7	2.89
3	2.33		8	2.93
4	2.67			
5	2.76		∞	π

For $\Gamma = 1/3$

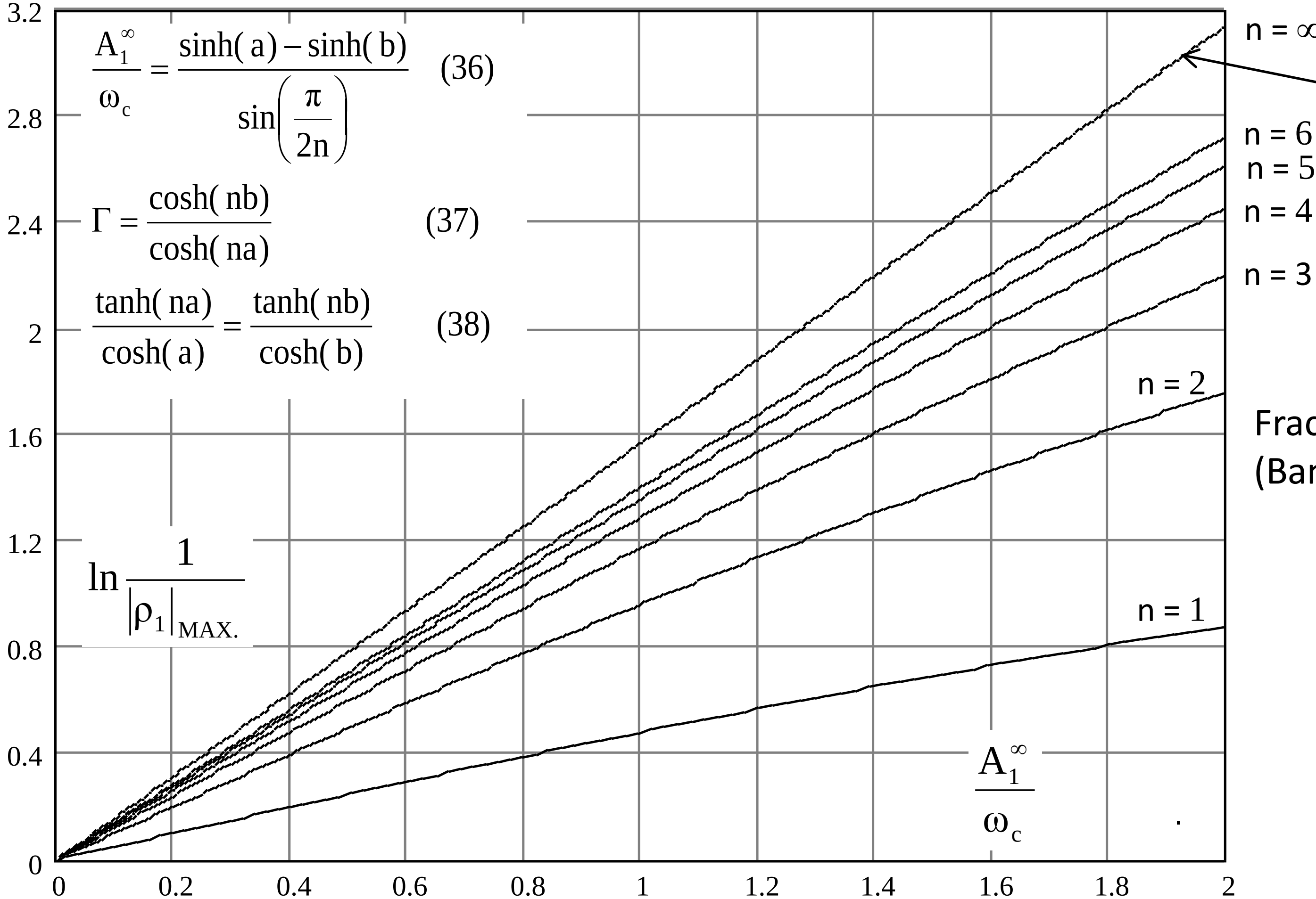
$\frac{B_2}{B_1} = 2.31$ (131% Increase)

$\frac{B_\infty}{B_2} = 1.65$ (65% Increase)

$\frac{B_3}{B_2} = 1.18$ (18% Increase ?)

Sent letter to Professor Fano asking for help in determining accuracy of a_n

1973 Fano's Reply



$$\ln \frac{1}{\Gamma} = \frac{\pi}{QB}$$

Fractional Bandwidth (Band-Pass) Conversion

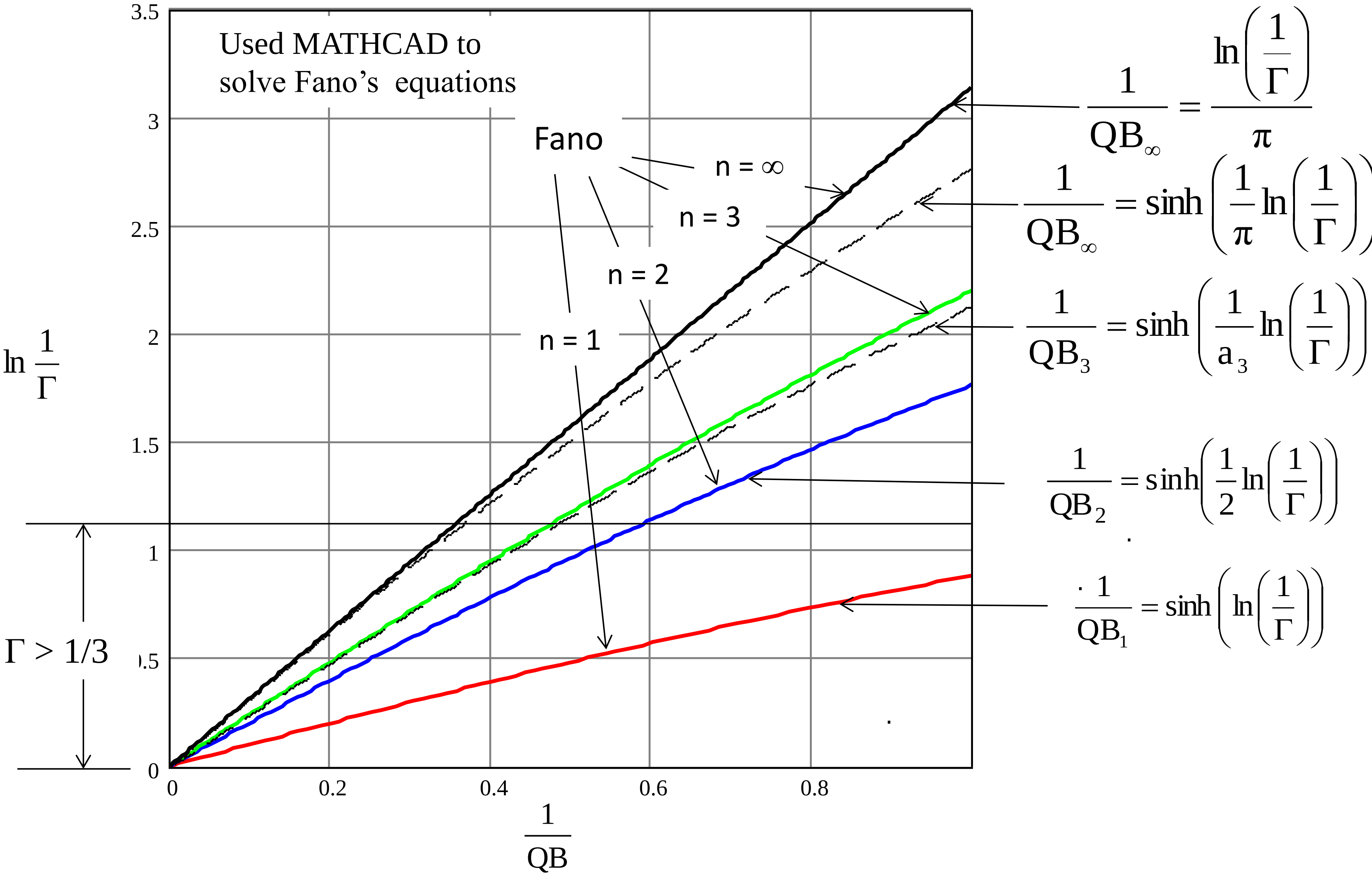
$$A_1^\infty = \frac{2R}{L}$$

$$\frac{A_1^\infty}{\omega_c} = \frac{2R}{\omega_0 L} \frac{\omega_0}{\omega_c}$$

$$\frac{A_1^\infty}{\omega_c} = \frac{2}{QB}$$

Fig. 19. Tolerance of match for a low-pass ladder structure with n elements

2004 – Comparison of Fano and Original Matching Equation



2004 Impedance-Matching Equation

$$B_n(\Gamma) = \frac{1}{Q} \frac{1}{b_n \sinh\left(\frac{1}{a_n} \ln\left(\frac{1}{\Gamma}\right)\right) + \frac{1-b_n}{a_n} \ln\left(\frac{1}{\Gamma}\right)}$$

b_n coefficient provides blending of the "sinh" and "ln" functions

$B_3/B_2 = 1.24$ (24% Increase)

Conclusion

- Wheeler's development of the principles for double-tuned impedance matching was a major contribution. Although it was developed for lumped-element circuits it has a broader application
- One can see by inspection that his solutions were optimum
- We have developed the Impedance-Matching Equation, a closed form solution for the Fano Equations, which we hope will be helpful and useful to the community
- What impressed me the most in all of this work was the remarkable fact that Wheeler's results, using the reflection chart, were identical to the results obtained by Fano using high-level network theory

Wheeler and Fano

Wheeler (Reflection Chart)

$n = 1,2$

$$B_n(\Gamma) = \frac{1}{Q}\frac{2\Gamma^{\frac{1}{n}}}{1-\Gamma^{\frac{2}{n}}}$$

Fano (Network Theory)

$n = 1,2,3\ldots.\infty$

$$B_n(\Gamma) = \frac{1}{Q}\frac{2\sin\left(\frac{\pi}{2n}\right)}{\sinh(a)-\sinh(b)}$$

$$\frac{\tanh(na)}{\cosh(a)} = \frac{\tanh(nb)}{\cosh(b)}$$

$$\frac{\cosh(nb)}{\cosh(na)} = \Gamma$$

$n = 1$

$$B_1(\Gamma) = \frac{1}{Q}\frac{2}{\frac{1}{\Gamma}-\Gamma}$$

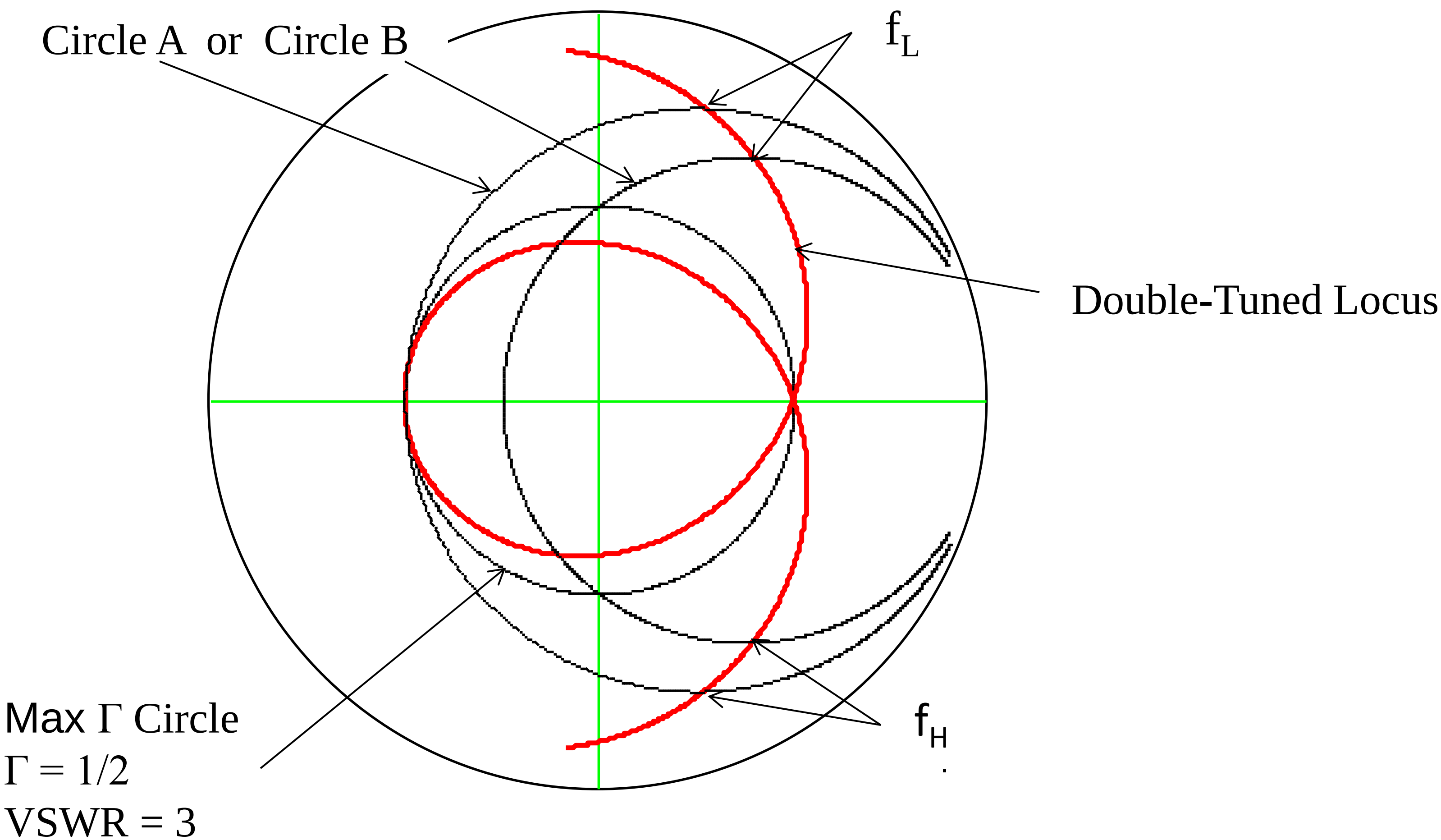
$$B_1(\Gamma) = \frac{1}{Q}\frac{2}{\sinh(a)-\sinh(b)}$$

$$\sinh(a) = \frac{1}{\Gamma} \qquad \sinh(b) = \Gamma$$

Triple-Tuned Impedance Matching

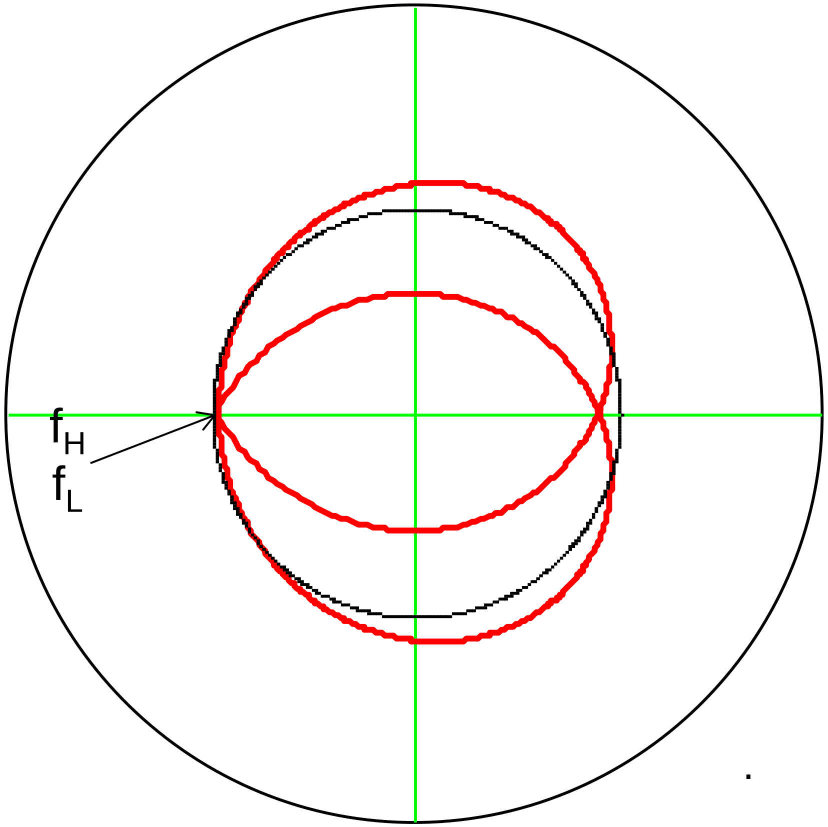
Triple-Tuned Impedance Matching

Which circle, A or B, should be used to position the edge-band frequencies on the Max Γ Circle

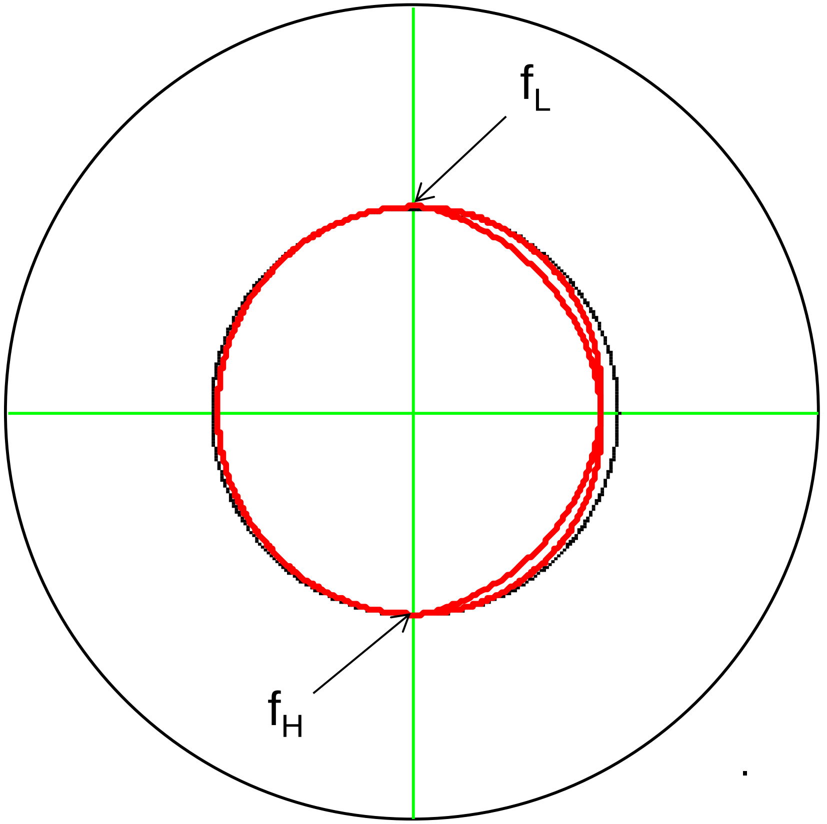


Triple-Tuned Impedance Matching Cont'd

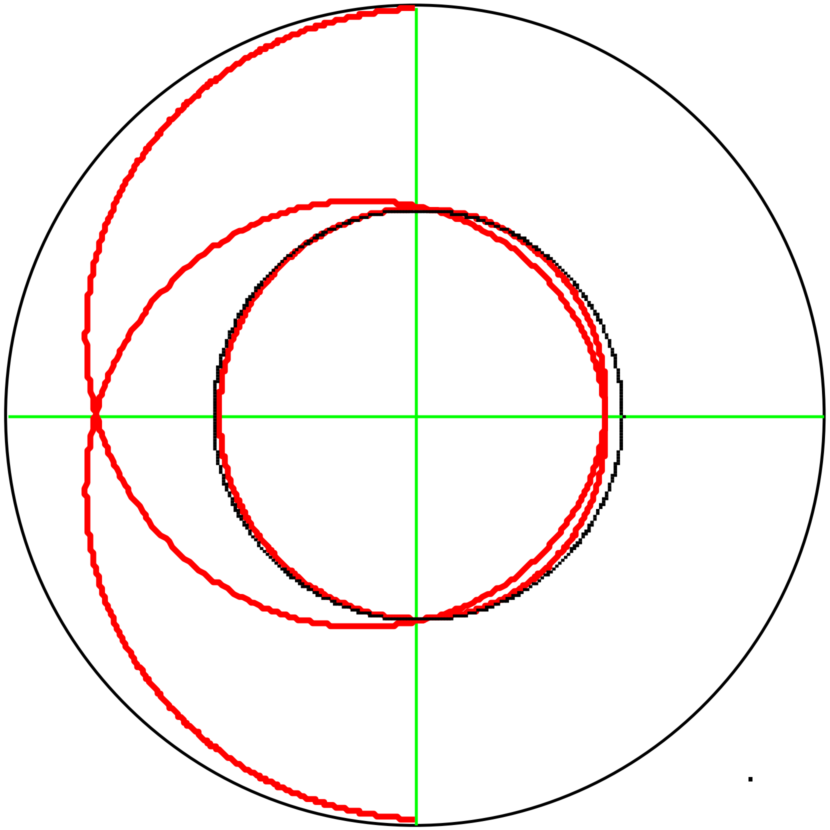
Edge-Band Frequencies
on Horizontal Axis



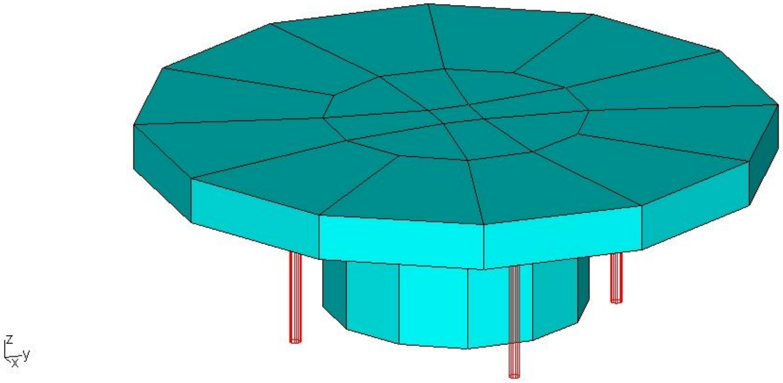
Edge-Band Frequencies
on Vertical Axis

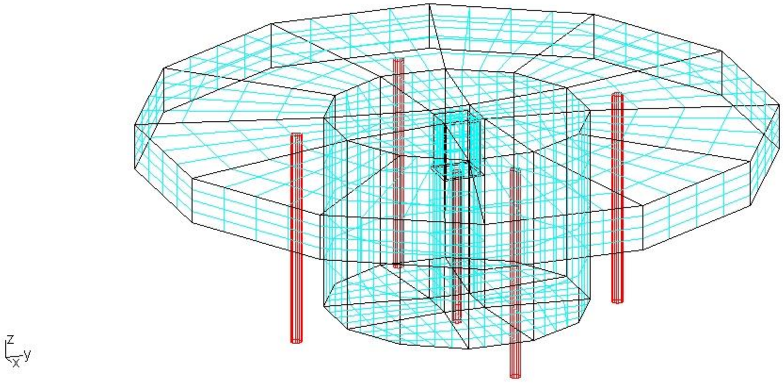


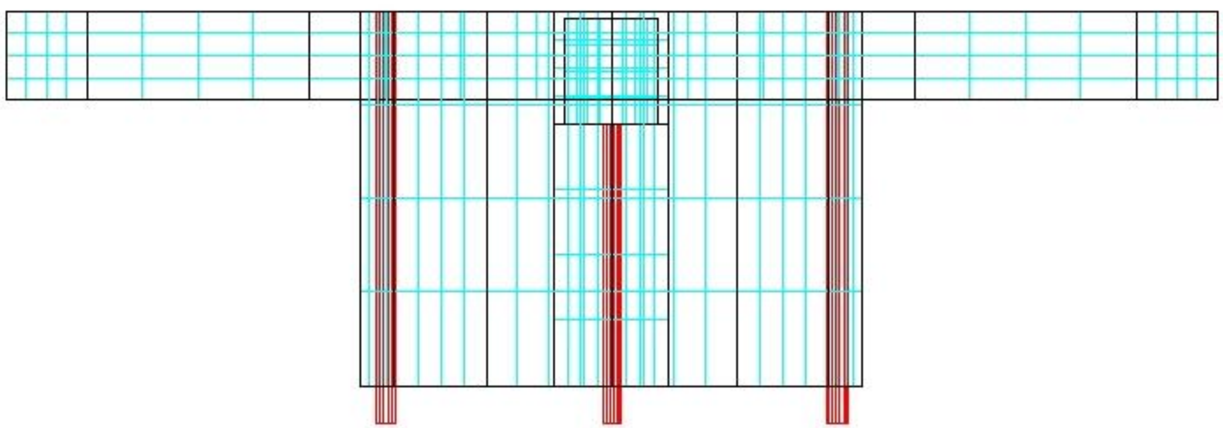
Triple-Tuned Impedance Matching Cont'd



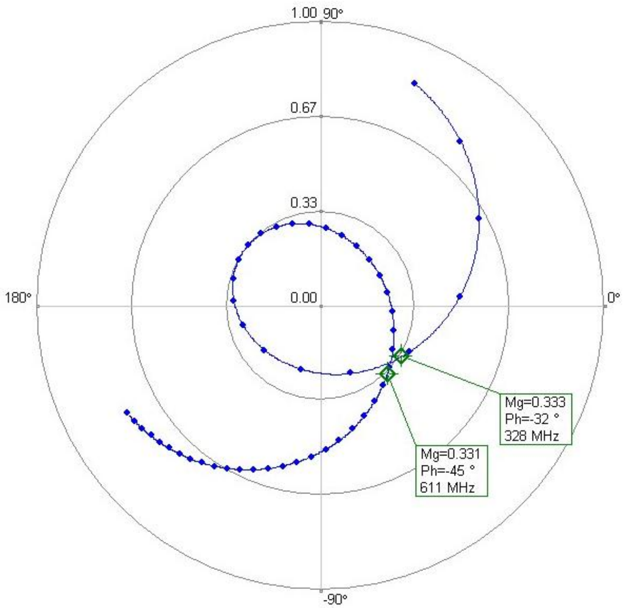
Triple-Tuned Monopole Antenna On Infinite Ground Plane





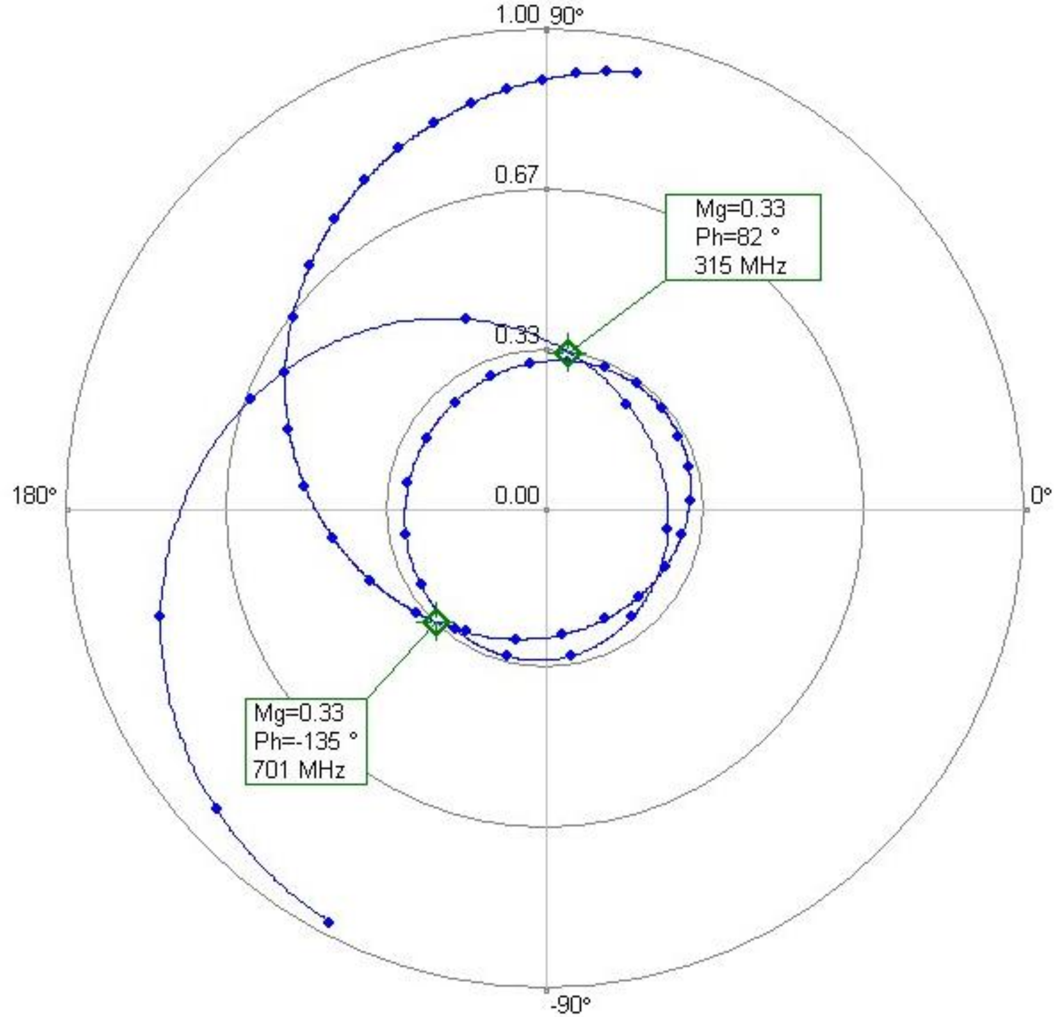


Triple-Tuned Monopole Antenna (Continued)



Double Tuned

Triple Tuned



Triple-Tuned Monopole Antenna (Continued)

Tuning	f_{Low} (MHz)	f_{High} (MHz)	f_0 (MHz)	B (Ratio)	B Increase (Percent)
Double	328	612	448	0.63	
Triple	315	703	470	0.83	32
				Theoretical	24