Extending Laplace and Fourier Transforms, and the Case of Variable Systems: A Personal Perspective

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Outline Of the Talk

- Dynamic System Representation
- Operator Calculus
- Transformation
- Laplace Transform and Complex-Time Systems
- An Example of A Complex-Time System - SSB
- Characterization of LTV Systems
- Operational Calculus in Two Variables
- Two-Dimensional Laplace Transform Techniques
- Representation of "*Time*" in System Theory
- A New Perspective

System Representation in Time Domain

- **The classical theory of variable systems is based on the solutions of linear ordinary differential equations with varying coefficients. The varying coefficients are usually functions of an independent variable, also called the *time variable*. The *time variable* is assumed to be *real* for physical systems.**

$$\sum_{i=0}^{n} a_i(t)\,x_i^{(i)}(t) = \sum_{k=0}^{m} b_k(t)\,y_k^{(k)}(t)$$

Operator Calculus Characterization

The fundamental (differential) equation of an LTV system is:

$$\sum_{i=0}^{n} a_i \frac{d^i y}{dx^i} = f(x) \qquad a_n = 1$$

Use the operator $D^i \longrightarrow \frac{d^i}{dx^i}$

The fundamental equation converts to: $\sum_{i=0}^{n} a_i D^i y = f(x)$

Use the operator $s \longrightarrow D$

The fundamental equation (for a system at rest) converts to:

$$\sum_{i=0}^{n} a_i s^i y(x) = f(x)$$

Using Operator Calculus

Observation 1 – As a result of using the operator calculus the homogeneous response has a *pattern*. The response of homogeneous equation:

$$\sum_{i=0}^{n} a_i s^i y(x) = 0$$

is a linear combination of exponentials:

$$y(x) = \sum_{i=0}^{n} a_i e^{s_i x}$$

s_i's are roots of the operator (characteristic) equation:

$$\sum_{i=0}^{n} a_i s^i = 0$$

Extending the Operator Calculus: Transformation

Expand the fundamental equation: $\sum_{i=0}^{n} a_i \frac{d^i y}{dx^i} = f_n(x) = \sum_{i=0}^{n} f(i\Delta x)\Delta x$

By generalization of the homogeneous solution, assume an exponential solution :

$$y(x) = e^{s_n x}$$

The fundamental equation yields:

$$\sum_{i=0}^{n} a_i s_n^i e^{s_n x} = \sum_{i=0}^{n} f(i\Delta x)\Delta x$$

where s_n is a root of the operator (characteristic) equation:

$$\lim_{n \longrightarrow \infty} \sum_{i=0}^{n} a_i s_n^i = \lim_{n \longrightarrow \infty} \sum_{i=0}^{n} f(i\Delta x) e^{-s_n x} \Delta x$$

$$\mathscr{L}\{f(x)\} = F(s) = \int_0^{+\infty} f(x) e^{-sx} dx$$

Laplace Transform: An Operational Calculus Tool

Introduced by Laplace in 1771 and applied (modern use) by Oliver Heaviside.
Observation 2 – The Laplace transform is obtained as a result of extending the concept of the operator calculus for solving differential equations, which can describe the fundamental equation of physical (dynamic) systems.
Observation 3 – The solution exists if there are finite numbers *M* and σ_0 such that :

$$|f(x)| < Me^{\sigma_0 x} \qquad \forall x \geq 0$$

Observation 4 – The independent variable x can represent any parameter (of the system); e.g., the “time.”
Observation 5 – **s** is the root of the characteristic equation $F(s) = 0$
Hence it is a complex number (or better said, a complex variable) in general.

Observation 6 – If x represents the independent time variable, then by definition, **s** represents the (complex) frequency in the *transform domain*.

Question 1 – Can *f(x)* be a complex function of *x*?
Question 2 – Can *x* represent an independent complex variable?

Example of A Complex Time System

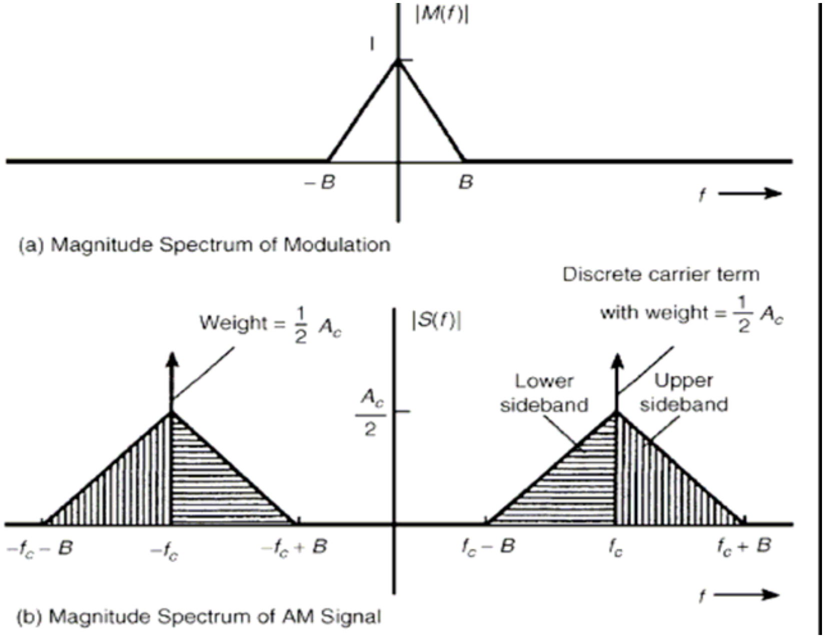
The single side band (SSB) amplitude modulation (AM) is an example of "complex-time" systems. The SSB spectrum is obtained by shifting the spectra

$$M_+(\omega) = M(\omega)u(\omega)$$

and

$$M_-(\omega) = M(\omega)u(-\omega)$$

by ω_c and $-\omega_c$ respectively, as shown.



Single Side Band System

It can be shown that: $m_+(t) = \frac{1}{2}\left[m(t) + jm_h(t)\right]$

$$m_-(t) = \frac{1}{2}\left[m(t) - jm_h(t)\right]$$

:

where $m_h(t)$ is the Hilbert transform of $m(t)$

$$m_h(t) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{m(\tau)}{t-\tau}d\tau$$

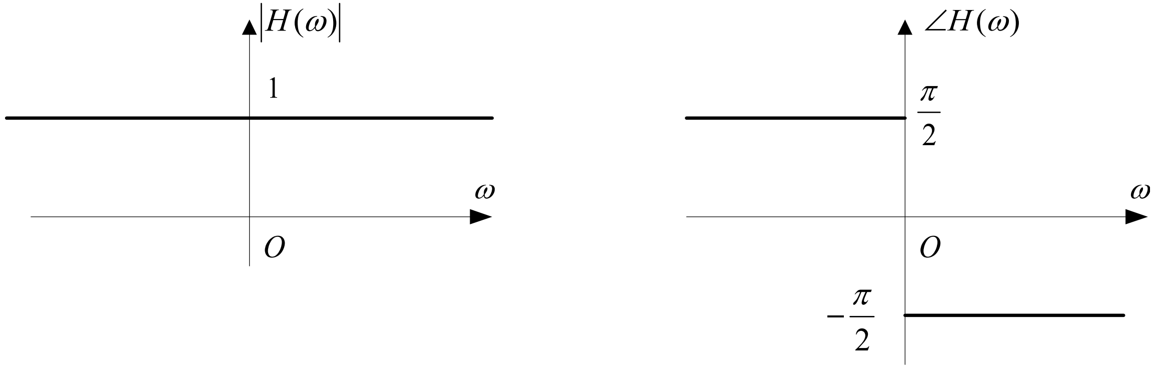
Fourier transform of $m_h(t)$ is: $M_h(\omega) = -jM(\omega)\,\text{sgn}(\omega) = M(\omega)H(\omega)$ where:

$$H(\omega) \stackrel{\Delta}{=} -j\,\text{sgn}(\omega) = \begin{cases} -j = e^{-j\frac{\pi}{2}} & \omega > 0 \\ j = e^{j\frac{\pi}{2}} & \omega < 0 \end{cases}$$

$H(\omega)$ is the transfer function of $m_+(t)$ an ideal $\frac{\pi}{2}$ phase shifter that

produces the imaginary-time part of the real-time function $\boldsymbol{m_+(t)}$.

SSB System (cont.)



Representation of the transfer function $H(\omega)$, an ideal $\frac{\pi}{2}$ phase-shifter.

The USB signal is:

$$\Phi_{USB}(\omega) = M_+(\omega - \omega_c) + M_-(\omega + \omega_c)$$

In time domain:

$$\varphi_{USB}(t) = m_+(t)e^{j\omega_c t} + m_-(t)e^{-j\omega_c t}$$

Substituting for $m_+(t)$ and $m_-(t)$ in this equation results in:

$$\varphi_{USB}(t) = m(t)\cos\omega_c t - m_h(t)\sin\omega_c t$$

SSB System: An Example

For $M(\omega) = 2\pi e^{-a|\omega|}$ find $m_+(t)$.

$$\mathscr{F}^{-1}\{M(\omega)\} = \mathscr{F}^{-1}\left\{2\pi e^{-a|\omega|}\right\} = \frac{2a}{t^2 + a^2}$$

The Fourier transform of the Hilbert transform of $M(\omega)$ is:

$$M_h(\omega) = -jM(\omega)\operatorname{sgn}(\omega) = -j2\pi\left[e^{-a\omega}u(\omega) - e^{a\omega}u(-\omega)\right]$$

The Hilbert transform is:

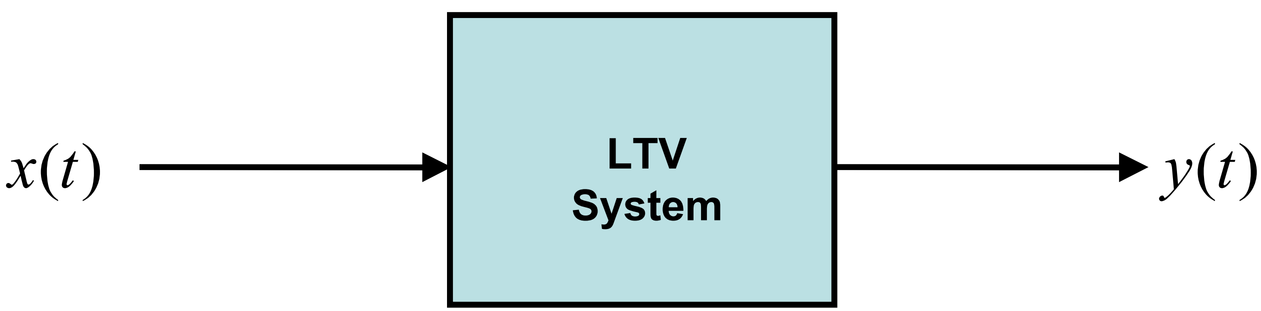
$$m_h(t) = \mathscr{F}^{-1}\{M_h(\omega)\} = -j\left[\frac{1}{a-jt} - \frac{1}{a+jt}\right] = \frac{2t}{t^2+a^2}$$

$$m_+(t) = \frac{1}{2}\left[m(t) + jm_h(t)\right] = \frac{a+jt}{t^2+a^2}$$

Characterization of Linear Time Varying Systems

- **Consider a single-input single-output (SISO) linear dynamic system characterized by the fundamental (differential) equation of a LTV system:**

$$\sum_{i=0}^{n} a_i(t) x_i^{(i)}(t) = \sum_{k=0}^{m} b_k(t) y_k^{(k)}(t)$$



Characterization of LTV systems (Cont.)

Characterization in Operator form:

$$\sum_{i=0}^{n} a_i(t)D^i y(t) = \sum_{i=0}^{n} b_i(t)D^i x(t)$$

$$L(D,t)y(t) = K(D,t)x(t)$$

where:

$y(t)$ = the output response signal

$x(t)$ = the input (excitation) signal

$a_i(t)$ = system variable parameter, known continuous function of time

$b_k(t)$ = system time-varying parameter, known continuous function of time

D_i = the ith differential operator (d^i/dt_i)

$L(\cdot,\cdot)$= the system output operator, known bivariate polynomial of time and differential operator

$K(\cdot,\cdot)$= the system input operator, known bivariate polynomial of time and differential operator

Observations on the LTV systems

Observation 1 – In general, time clocks of the *signal* and *system* does not have to be synchronized; i.e., the (time) variables of the signal and systems can be *independent* of each other.

$$L(D,\tau)y(t) = K(D,\tau)x(t)$$

Observation 2 – At any instant of *"t"* there is a response, which is a specified function of "τ".

Observation 3 – At any fixed "τ" there is a response, which is a specified function of "t".

Observation 4 – The system response is a function of variations of observation parameter "t" and application parameter "τ".

Observation 5 – A zero-input, SISO LTV system described by:

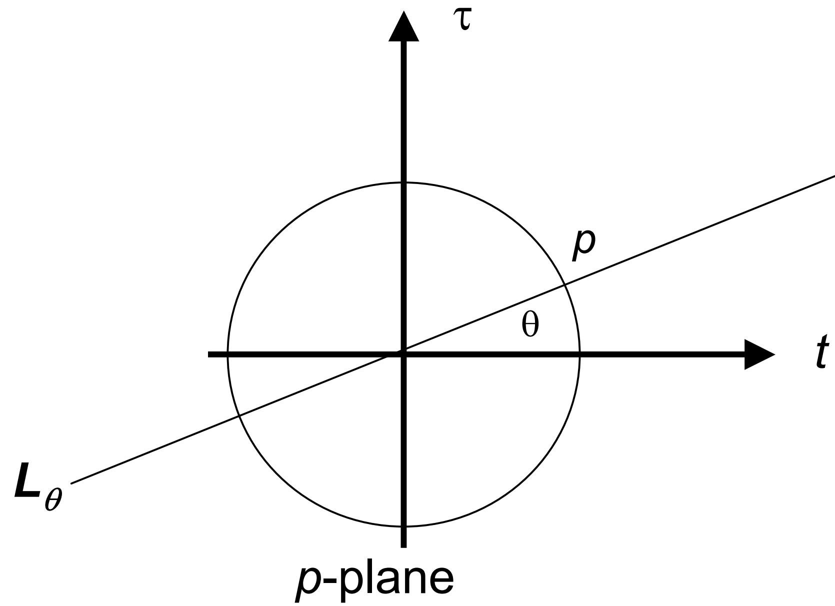
$$L(D,\tau)y(\cdot) = 0$$

is a linear system that its natural frequencies are varying with "τ". In other words, the solutions of this equation are exponential functions of time with varying natural frequencies, as given by:

$$y(\cdot) = \sum_{i=0}^{n} c_i e^{-t\alpha_i(\tau)}$$

where $\alpha_i(\tau)$ is a function of variable coefficients of the fundamental equation of the system under consideration.

Extension to Operator Calculus



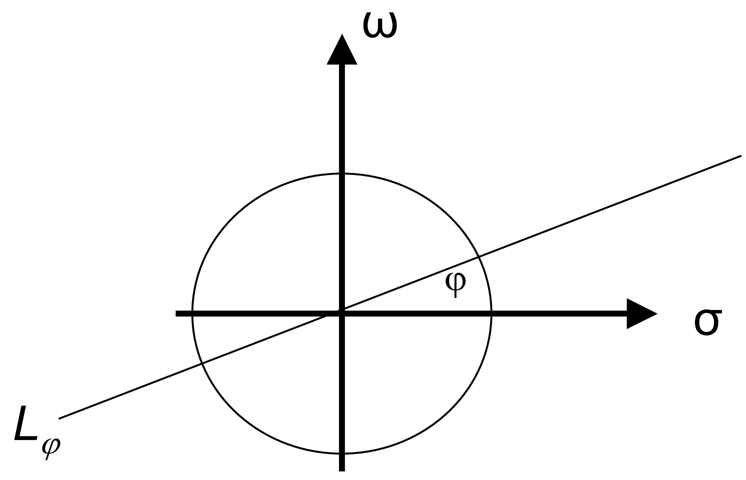
Observation 6 – We replace the real *t* by a complex *p* in a given *f(t),* as the (time) variables of the signal and systems can be *independent* of each other. Thus, define a complex plane *p-plane*, as $p = t + j\tau$.

Observation 7 – This is equivalent to converting a 1-D real variable function to a 2-D real variable function (function of a complex variable).

Observation 8 – This is equivalent to converting a 1-D real variable function to a 2-D real variable function (function of a complex variable).

Observation 9 – The line denoted as L_θ passes through the origin making an angle θ with the real axis on *p*-plane.

Time-Frequency Representation By Operator Calculus



s-plane

Observation 10 – The corresponding complex variable in the transform domain is $s = \sigma + j\omega$, and represents the variations in the *s*-plane.

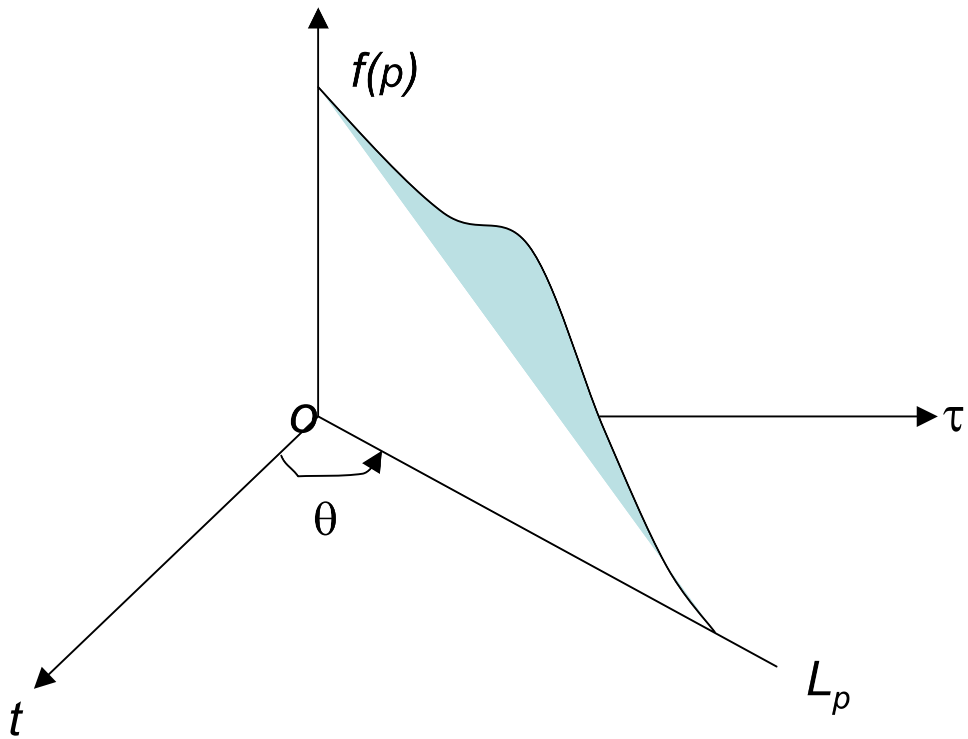
Observation 11 – The extended 2D version of the Fourier transform, $\mathcal{F}_{2D}$, of *f(t)* (which can also be viewed as the 2DLT) is:

$$F_{2D}\{f(t)\}|_{p,s} = \int_{L\theta} f(p)e^{-sp}dp$$

Observation 12 – The line L_θ gives a 1-D profile of this 2-D function. The angle θ in the common definition of 1-D transform is zero.

Observation 13– The F_e is also a 2-D real variable function. The 1-D projection over $L\varphi$ is the common 1-D Fourier transform over L_θ provided $\theta + \varphi = \pi/2$.

Real Variable Function Representation in *p*-Plane

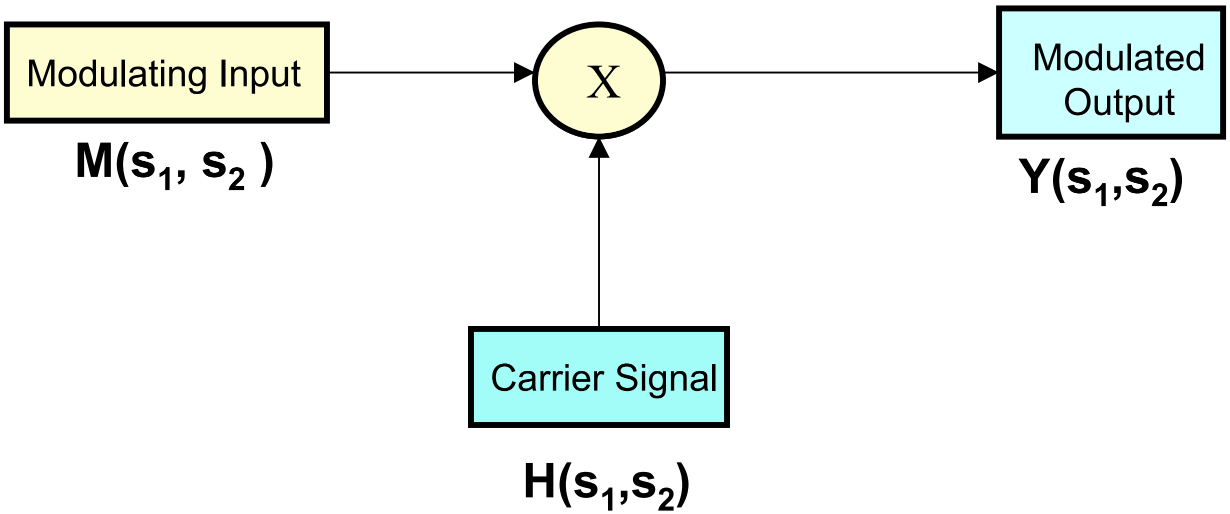


Observation 14 – The common transform and its inversion are evaluated by integral over *t-axis* (pure real) and *ω-axis* (pure imaginary), respectively.

Observation 15 – We adopt the two-sided (bilateral) Laplace and Fourier transform.

$$\mathrm{L}\{f(t)\} = F(s) = \int_{-\infty}^{+\infty} f(t)e^{-st}dt$$

Operational Calculus in Two Variables



Observation 15 – The operational calculus in two variables, which is a generalization of the classical Laplace transform, was introduced during 1930 [P. Humbert, J.C. Jaeger], but appears to have remained largely unknown to the signal processing community (especially, for analog signal processing).

Observation 16 – The potential application of operational calculus in two variables to analysis of variable characteristic communication channels and systems has been realized.

Extension of the Operator Calculus to Solution of LTV Systems

Observation 6 – Considering the invariance property of

$$L(D,\tau)y(t) = K(D,\tau)x(t)$$

with respect to "t" and "τ", and by analogy with the case of LTI systems, we interpret this equation as a two-dimensional system model, and shall use a two-dimensional operator calculus (i.e. two-dimensional Laplace transform (2DLT)) to find its response.

$$\mathscr{L}_{2D}\{L(D,\tau)y(t)\} = \mathscr{L}_{2D}\{K(D,\tau)x(t)\}$$

$$L(s_1,s_2)Y(s_1) = K(s_1,s_2)X(s_1)$$

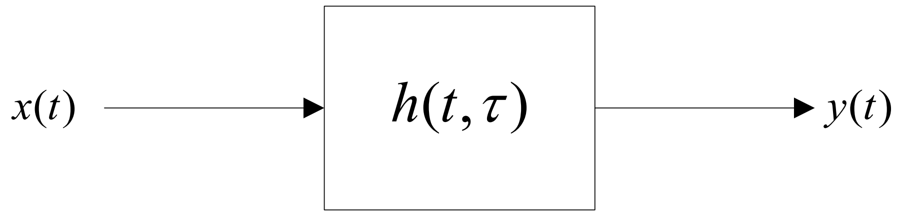
$$Y(s_1) = \frac{K(s_1,s_2)}{L(s_1,s_2)}X(s_1)$$

where:

$$K(s_1,s_2) = \sum_{i=0}^{n} B_i(s_2)s_1^i$$

$$L(s_1,s_2) = \sum_{i=0}^{n} A_i(s_2)s_1^i$$

2DLT Solution of LTV Systems



If $x(t,\tau)=\delta(t,\tau)=\delta(t)\delta(\tau)$ we denote the 2D transfer function, $H(s_1,s_2)$ of an LTV system as:

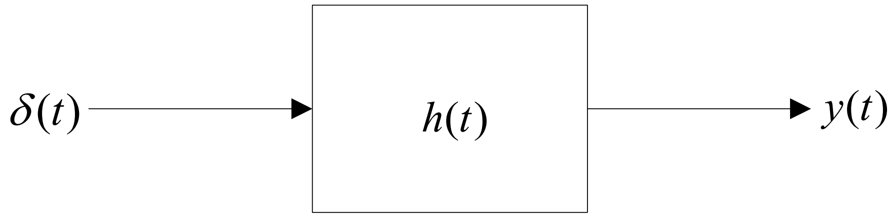
$$H(s_1,s_2)=\frac{K(s_1,s_2)}{L(s_1,s_2)}=\sum_{i=0}^{n}\frac{B_i(s_2)s_1^i}{A_i(s_2)s_1^i}$$

Where:

$$h(t,\tau)=\mathscr{L}^{-1}{}_{2\mathscr{D}}\{H(s_1,s_2)\}=\frac{1}{(2\pi j)^2}\int_{\sigma_2-j\infty}^{\sigma_2+j\infty}\int_{\sigma_1-j\infty}^{\sigma_1+j\infty}H(s_1,s_2)e^{s_1 t}e^{s_2\tau}ds_1 ds_2$$

$H(s_1,s_2)$ and $h(t,\tau)$ are called the bi-frequency transfer function and bivariate impulse response, respectively.

Special Case: LTI System



In the case of LTI systems, coefficients a_i and b_i are all constants; $H(s_1, s_2)$ reduces to the familiar transfer function

$$H(s) = \frac{K(s)}{L(s)} = \sum_{i=0}^{n} \frac{b_i s^i}{a_i s^i}$$

Letting $s_1 = s_2 = j\omega$ Will result in an inverse of the two-dimensional Fourier Fourier Transform of $H(j\omega, j\omega)$ as given by:

$$h(t,\tau) = \frac{1}{(2\pi)^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} H(j\omega, j\omega) e^{-j\omega(t+\tau)} d\omega d\omega$$

Representation of "*Time*" in System Theory

- The "time variable" is assumed to be a *real* variable for physical systems. This assumption facilitates analysis and synthesis of fixed (time-invariant) systems by allowing the *Laplace transform* techniques to be used.
- However, the assumption of "real time" is shown to be inadequate for realization of time-varying systems in the *transform domain.*

A New Perspective

- The discussion in this presentation is based on a different point of view.
- Possibility of system realization through an examination of the behavior of systems that are functions of a *complex* time-variable.
- This approach allows, in effect, a two-dimensional transform techniques to be used for the time-varying systems in the same manner that the conventional frequency-domain techniques are used in connection with fixed systems.

Conclusion

- We have presented the extension of the Fourier and Laplace transform techniques, which is designated bivariate operational calculus.
- The two-dimensional transform techniques provide a useful framework for some applications.
- The discrete-time version of two-dimensional Laplace transform (2DLT) has already been used in digital signal processing.
- The relationship between the 2DLT and time-frequency transform suggest that it can be a useful tool for analysis and synthesis of linear time-varying systems.
- The work presented here opens several areas for further investigations.